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A Role for Attentional Reorienting During Approximate Multiplication and Division

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Abstract

When asked to estimate the outcome of arithmetic problems, participants overestimate for addition problems and underestimate for subtraction problems, both in symbolic and non-symbolic format. This bias is referred to as operational momentum effect (OM). The attentional shifts account holds that during computation of the outcome participants are propelled too far along a spatial number representation. OM was observed in non-symbolic multiplication and division while being absent in symbolic multiplication and division. Here, we investigate whether (a) the absence of the OM in symbolic multiplication and division was due to the presentation of the correct outcome amongst the response alternatives, putatively triggering verbally mediated fact retrieval, and whether (b) OM is correlated with attentional parameters, as stipulated by the attentional account. Participants were presented with symbolic and non-symbolic multiplication and division problems. Among seven incorrect response alternatives participants selected the most plausible result. Participants were also presented with a Posner task, with valid (70%), invalid (15%) and neutral (15%) cues pointing to the position at which a subsequent target would appear. While no OM was observed in symbolic format, non-symbolic problems were subject to OM. The non-symbolic OM was positively correlated with reorienting after invalid cues. These results provide further evidence for a functional association between spatial attention and approximate arithmetic, as stipulated by the attentional shifts account of OM. They also suggest that the cognitive processes underlying multiplication and division are less prone to spatial biases compared to addition and subtraction, further underlining the involvement of differential cognitive processes.

Keywords: operational momentum, approximate calculation, spatial attention, mental number line, reorienting, mental arithmetic

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Numerical cognition has long been thought to represent a prime example of an abstract propositional symbolic system with no obvious reference to the outer world. For example, number names such as ‘one’, ‘two’ and so on do not bear any obvious association to the referenced cardinal meaning. Recent evidence, however, implies that numbers and mental arithmetic bear numerous associations with space and time (Fischer & Knops, 2014; Hubbard, Piazza, Pinel, & Dehaene, 2005). Beyond using space to represent numerical magnitude along a spatially oriented mental number line (MNL), it has been hypothesized that humans re-use their spatial prowess and the underlying neural resources to master numerical operations (Dehaene & Cohen, 2007; Knops, Thirion, Hubbard, Michel, & Dehaene, 2009). The cortical mechanisms in parietal cortex that serve for the transformation of spatial coordinates in different referent frames (Beck, Latham, & Pouget, 2011; Pouget, Deneve, & Duhamel, 2002) may play a pivotal role in this context. In particular, neural populations in posterior superior parietal cortex encode object location with respect to different body parts and references. For example, during the planning of eye movements these circuits compute the head-centred position of a given object by adding the activity of neural populations that encode eye-centred coordinates and eye position (Beck et al., 2011; Pouget et al., 2002). Hence, the cortical algorithms for executing simple additions are present in parietal cortex. Basic mathematical operations might co-opt these circuits by providing numerical input instead of coordinate information (Dehaene & Cohen, 2007; Hubbard et al., 2005; Knops, Thirion, et al., 2009).

This process is not bias-free. When approximating the outcome of simple addition or subtraction problems humans are likely to provide a biased response that deviates systematically from the correct outcome. The results of addition problems tend to be overestimated while the results for subtraction problems are underestimated. This cognitive bias is referred to as operational momentum (OM; McCrink, Dehaene, & Dehaene-Lambertz, 2007) due to its resemblance to a systematic bias in the estimation of the point in space where a moving object disappeared (Hubbard, 2005, 2015). OM has initially been observed with non-symbolic operations where participants estimated the number of visual objects (dots) that would result from adding two sets of dots or subtracting one set from the other (McCrink et al., 2007). However, OM has also been found in symbolic notation (Knops, Dehaene, Berteletti, & Zorzi, 2014; Knops, Viarouge, & Dehaene, 2009) which has been interpreted as evidence for a common underlying mechanism. OM has also been observed with paradigms that require translating the cognitively generated numerical estimate into a position on a labeled line (Pinhas & Fischer, 2008), or actively producing the outcome via a dot generating manual device (Lindemann & Tira, 2011). In the labeled line task, participants indicated the position of the outcome of visually presented addition and subtraction problems on a line that was labeled with zero on the left and 10 on the right. Crucially, participants misplaced addition outcomes to the right and subtraction outcomes to the left, compared to baseline that required indicating the numbers' positions without preceding arithmetic operation. When actively producing dot patterns corresponding to the outcome of multi-digit problems, participants produced relatively larger estimates for addition problems as compared to subtraction problems with identical correct outcome (Lindemann & Tira, 2011). Together, this implies that mental arithmetic is subject to systematic biases that may have their origin in the application of spatial coordinate transformation mechanisms to numerical quantity information that can be conceived of as positions on the MNL.

More evidence for spatial contributions to mental arithmetic comes from a recent study reporting systematic interference between arm or eye movements and mental arithmetic (Wiemers, Bekkering, & Lindemann, 2014). Addition performance was impaired when participants moved their arms or eyes downward while subtraction was affected with upward movements of the arm or eye. In horizontal plane only arm but not eye movements to the left interfered with addition while arm movements to the right interfered with subtraction. During simple addition and subtraction tasks, Margethis and colleagues found systematic deviations of manual mouse pointer trajectories from the ideal path such that addition and subtraction trajectories deviated to the right and left from the ideal paths, respectively (Marghetis, Núñez, & Bergen, 2014). Interestingly, this effect was observed over and above confounding the response position with numerical magnitude (Pinhas & Fischer, 2008; Pinhas, Shaki, & Fischer, 2014) because the correct outcomes were presented both to the upper left or right relative to starting position.

Despite the growing number of studies demonstrating OM in different settings, the underlying mechanisms of the OM effect are currently debated. Three major hypotheses vie with one another. First, it has been suggested that the OM effect reflects the outcome of a simple heuristic that would associate different arithmetic operations with expectations concerning the numerical relationship between the outcome and the operands (McCrink et al., 2007). For addition, this heuristic would predict that the result should be larger than either of both operands. For subtraction, the heuristic would predict outcomes that are smaller than the first operand. The same heuristic would hold for other arithmetic operations such as multiplication and subtraction. In particular, the observed OM in 9-month-olds supports this account because it is unlikely that at that age infants have developed a spatial representation of numerical magnitude (McCrink & Wynn, 2009). A second hypothesis assumes that OM results from the flawed logarithmic compression and decompression into a linear metric during the arithmetic process (Chen & Verguts, 2012). The mental number line is assumed to be logarithmically compressed (Dehaene, 2001; Nieder & Dehaene, 2009). The compression and decompression processes needed to transduce between linear and logarithmic scales may be flawed. In the extreme variant of the compression hypothesis, addition and subtraction operate on compressed values, that is, the decompression fails entirely. This would result in massive over- and underestimations since the sum of the logs is the log of a product (i.e. $\log(a) + \log(b) = \log(a \times b)$). An equivalent underestimation would result for subtraction since the difference of the logs is the log of a quotient (i.e. $\log(a) - \log(b) = \log(a/b)$).

Finally, spatial accounts have been proposed to account for OM. According to the spatial competition hypothesis (Pinhas & Fischer, 2008; Pinhas, Shaki, & Fischer, 2015), OM is the result of the “competing spatial biases invoked by the operands, the operation sign, and the result of an arithmetic problem” (p. 997; Pinhas et al., 2015). Operands and results activate their respective positions on the mental number line, that “compete for responses” (p. 413; Pinhas & Fischer, 2008). For subtractions, for example, the result of a given problem may be located between the operands (e.g. 7 - 2 = 5) or to the left of both operands (e.g. 7 - 4 = 3). Compared with problems involving zero as a second operand, these competing biases mitigate the observed bias towards the result which is more pronounced when the second operand does not additionally compete for responses. Consequently, Pinhas and Fischer (2008) observed the largest OM bias with zero problems. Recently, these authors observed an inverse OM when reversing the line labels such that the right end of the line was labeled with 0 and the left end with 10 (Pinhas et al., 2015), providing support for their spatial competition bias and underlining the observation that number-to-space mappings are highly flexible (Bächtold, Baumuller, & Brugger, 1998). According to a second spatial account, the OM reflects systematic biases from the deployment of the coordinate transformation system in parietal cortex that also mediates attentional shifts in space (Knops et al., 2014; Knops, Thirion, et al., 2009; Knops, Viarouge, et al., 2009; Knops, Zitzmann, & McCrink, 2013). According to this approach, approximate mental arithmetic is mediated by a dynamic interaction between positional codes on the MNL and an attentional system that shifts the spatial focus to the left or right. At the neural level this may be instantiated in the functional interactions between areas along the intraparietal sulcus and posterior, superior parietal areas. The idea is that a parietal circuit that has been proposed to combine retinal and eye position information via vector addition in order to compute positions in space may be recycled to implement mental arithmetic. The resulting positional information can be used to guide eye or hand movements and has been proposed to be the base for shifts of spatial attention. This places mental arithmetic in the realm of dynamic updating processes of spatial coordinates in parietal cortex and stipulates that the efficiency of this system is linked with arithmetic performance. Due to the approximate nature of this process the shifts may ‘overshoot’, leading to over- and underestimation in addition and subtraction, respectively. Interestingly, the latter approach suggests a functional coupling between eye movements and arithmetic. A recent study provided confirmatory evidence for this notion (Klein, Huber, Nuerk, & Moeller, 2014). Participants’ eye movements after the first saccade were observed to move to the right during addition problems and to the left in subtraction problems when asked to indicate the location of the result on a labeled line (but see Hartmann, Mast, & Fischer, 2015; Klein et al., 2014). A second implication from this account is that OM is not limited to addition and subtraction but generalizes to basic arithmetic transformations with natural numbers such as multiplication and division as long as they require quantity manipulations that are associated with attentional shifts along the MNL. Operations that lead to larger outcomes would be associated with rightward shifts of attention, while the opposite should hold for operations that lead to smaller outcomes. Katz and Knops (2014) investigated the OM effect in the context of multiplication and division. However, as opposed to simple addition and subtraction problems, finding the solution of simple symbolic multiplication problems is often conceptualized as verbally mediated recall from long term memory (e.g. Campbell & Xue, 2001), mixed with idiosyncratic short cuts (e.g. retrieving 9 × 7 = 63 by subtracting 7 from 70). Nevertheless, approximate estimates of the outcome might involve spatial transformations, protecting us against accepting grossly wrong solutions. For example, knowing that 32 × 8 must be a three-digit number helps excluding 40 as a possible outcome. Despite this possible role of spatial transformations in multiplication and division, neither for the standard set of multiplication tables (i.e. between 1 × 1 = 1 and 9 × 9 = 81), nor for two-digit × one-digit problems (e.g. 14 × 3 = 42) an OM effect was observed (Katz & Knops, 2014). However, for the corresponding non-symbolic problems where the quantities were presented as dot patterns a regular OM effect was observed. One crucial difference between the procedure adopted by Katz and Knops (2014) and previous studies (Knops, Viarouge, et al., 2009) was that the correct solution was presented as one of five response alternatives, potentially encouraging the engagement in exact calculation and direct retrieval rather than approximating the outcome. This may have reduced the opportunity to detect any systematic biases due to attentional shifts that accompany approximate quantity manipulations.

The aim of the current study was two-fold. First, we aimed at testing the presence of OM in multiplication and division by eliminating the presence of the correct outcome amongst the response alternatives. By encouraging participants to approximate even in the symbolic notation we aimed at increasing sensitivity to detect any systematic biases during multiplication and division with Arabic digits. Second, engaging participants in both an OM task and a Posner paradigm allowed us to test whether potential OM biases actually correlate with attentional measures. According to the above theoretical accounts of the OM we can break this question down into four aspects. Do attentional parameters correlate with (a) a heuristic according to which multiplication leads to larger outcomes and division to smaller outcomes, (b) flawed decompression, (c) competing spatial biases by the operands, the results or the outcome, or (d) attentional shifts along the mental number line? According to the heuristics account and the compression-decompression approach, no correlation with attentional measures would be predicted. Among the spatial accounts, only the attentional explanation predicts a correlation between attentional parameters and OM effect. No such correlation is predicted by the competing spatial biases account.

Methods

Participants

Participants (*n* = 18; 19-74 years-old, *M* = 36.3, *SD* = 19; 14 female, 4 male) were recruited from the general population (both student and non-student) using a departmental database. Written informed consent was obtained from all participants prior to participation. Students received course credit; no other reimbursement was given.

Procedure and Materials

The study consisted of two experiments; a calculation task involving symbolic and non-symbolic multiplication and division problems, and a variant of the Posner task to test different aspects of visuo-spatial attention (orienting/selection and reorienting/executive attention).

Calculation Task

The calculation task was created and presented using OpenSesame (Mathôt, Schreij, & Theeuwes, 2012). Participants were given written instructions and then performed 24 (12 symbolic, 12 non-symbolic; intermixed) practice calculation trials. Symbolic (Arabic digits) and non-symbolic (dot-arrays) multiplication and division problems were presented horizontally (separated by 'x' or '÷') to reduce working memory demands, for 2000ms, followed by seven response choices presented in a circle until a response was made or for a maximum of 6000ms (Figure 1). Compared to previous studies (Katz & Knops, 2014), we reduced presentation time of the operands to minimize propensity of adopting a counting strategy. Responses were made with a mouse-click on the chosen value or dot-array.

Non-symbolic stimuli were created using MATLAB and the Psychophysics Toolbox extension (Brainard, 1997; Pelli, 1997). To prevent that participants relied on non-numeric stimulus features such as density, occupied area or individual dot size we de-correlated quantity from visual parameters (area subtended, average dot-size) in each presented set of response alternatives using the method described by Gebuis and Reynvoet (2011). This method resulted in trial-specific response sets (7 dot-arrays) with no correlations between quantity, average dot size or area subtended ($-.2 < r < .2$ respectively).

To catch random responding, symbolic and non-symbolic control trials (16 ÷ 1, 16 x 1) were intermixed with calculation trials. Participants whose performance deviated more than 3 *SD* from the group mean or with symbolic control problem accuracy below 50% (chance = 14.3%) were excluded from the study.

To control for response choice magnitude effects, the same result values (with a random jitter in symbolic trials) or quantities (non-symbolic trials) were presented for multiplication and division; this resulted in different operands for multiplication versus division. We created a geometric series of 11 values ranging from 1/3 to three times the correct value (Knops, Viarouge, et al., 2009; Knops et al., 2013) for non-symbolic notation. For problems presented in symbolic notation the geometric series spanned values ranging from ½ times to two times the correct value. Arithmetic problems with symbolic range are reported in Appendix A. Because previous findings suggested that presenting the exact correct value may have made responses too accurate to detect symbolic response bias (Katz & Knops, 2014), symbolic response choices were jittered so that the correct answer was never presented (Knops, Viarouge, et al., 2009; Knops et al., 2013). To achieve this, all results were jittered either up or down by a random value which fell within plus or minus half the numerical interval between the correct result and the first deviant above or below it. The random value was drawn from a flat distribution on a logarithmic scale with a mean value of zero and was fixed for a given trial.

Participants completed 576 calculation trials (144 per condition) and 120 control trials (16 ÷ 1, 16 x 1; 30 per condition). To prevent the correct result always being the median value (*i.e.* 4th response value rank of 7

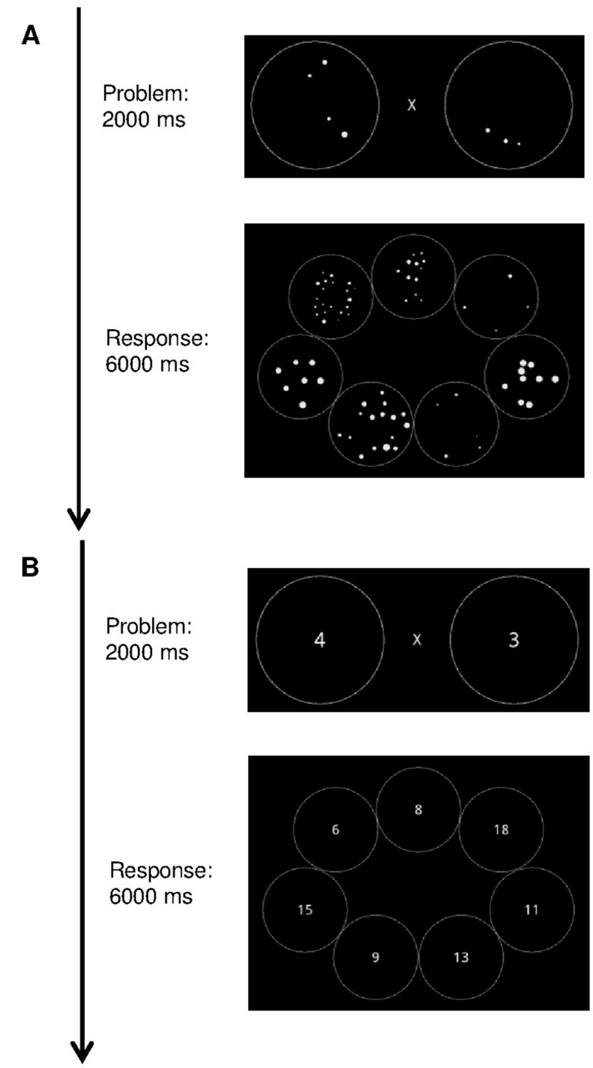

Figure 1. Calculation task. Symbolic and non-symbolic multiplication and division problems were presented in random order. Participants selected the responses with a mouse. (A) Non-symbolic multiplication. A set of 7 dot- arrays were created so that neither area subtended nor average dot-size correlated with quantity. (B) Symbolic multiplication. Response choices were jittered so that the exact answer was never shown.

choices), we varied the range of response alternatives and presented only seven out of the eleven response alternatives for each problem. The seven response alternatives corresponded to the smallest (low), largest (high) or middle (middle) range of response alternatives (Figure A.1). This means that the response alternative which is closest to the correct outcome and upon which the response alternatives were centred changed its ordinal rank for the different ranges. That is, for the smallest range, it fell close to the upper end (rank six out of seven), while it was located close to the lowest end for the highest range (rank two out of seven). In calculation trials, each problem was presented six times (two per range). At the end of the calculation task, participants were asked to describe how they solved the problems (“Please describe how you solved the problems. Which strategies did you use to decide for one of the response alternatives?”). Responses were typed in a blank document and no word limit was given. Strategy and word count were extracted from the responses.

Attention Task

Attention was assessed using an endogenous Posner cueing task (Posner, 1980), created and presented with MATLAB and the Psychophysics Toolbox (PTB) extension (Brainard, 1997; Pelli, 1997). Participants were seated approximately 60cm from the display and instructed to fixate on a red fixation dot (0.6°), which was presented at the center of the screen, flanked by a dark grey square box on either side (boxes: 3.3°, border width: 0.2°; see Figure 2). A yellow left, right, or double-ended (neutral cue) arrow (length: 2°) was presented in the center of the screen. A white target circle (2°) was flashed inside the box for 10ms, either on the side indicated by the arrow (valid cue), the opposite side (invalid cue) or randomly (neutral cue). Distance between fixation and target area border was 7.3°. The fixation sign without cue was presented for a variable duration between 800 and 1000 ms (M = 903.45ms, SD = 58.54ms). Cues were presented for a random duration between 1200 and 1800 ms (M = 1487ms, SD = 174.72ms). Participants were instructed to indicate the appearance of a target via button press (space bar) as quickly as possible. Participants were given a maximum of 1200ms to make a response. Participants completed 120 trials (84 valid (70%), 18 invalid (15%), 18 neutral (15%)).

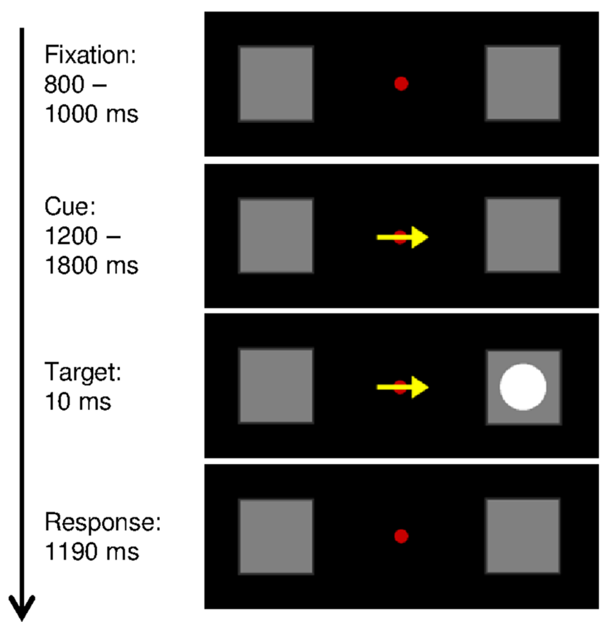

Figure 2. Attentional cueing task. An arrow appeared that either pointed in the direction of the subsequent target (valid cue, shown), the opposite direction (invalid cue) or in both directions (double-headed arrow, neutral cue). Participants were instructed to press the space bar as quickly as possible when they saw the target (white disk) that appeared after a variable SOA (1200 ms – 1800 ms).

Data Preparation and Analysis

Calculation task — Symbolic catch trials were always 16 multiplied or divided by 1. Although responses were jittered, so that '16' was never presented, if participants were paying attention and following the task directions, they should have been able to choose the value closest to 16 most of the time. Some degree of inaccuracy (i.e. < 100%) was expected since response choices were jittered. Therefore, we first eliminated subjects with accuracy less than 50% correct or greater than 3 SD from the group mean. This cut-off eliminated one subject whose accuracy (40%) deviated more than 3 SD from the group mean. In the remaining subjects (n = 17), the average accuracy was between 93% and 70% (M = 85%, SD = 0.36).

Next, because multiplication and division problems were not presented in separate blocks (Katz & Knops, 2014) and subjects might occasionally perform the wrong operation (*e.g.* multiplication instead of division), data was trimmed to exclude trials where the difference between the $\log_{10}$ of the chosen value and the $\log_{10}$ of the correct value was more than 3 SD from the subject's mean. When considering all conditions together, this excluded 40 calculation trials (0.4% of calculation responses, all non-symbolic). This was less than previous studies using this method (*e.g.* 1.8%; Knops, Viarouge, & Dehaene, 2009), possibly due to the simultaneous presentation of operands and the use of operation symbols (x, ÷) rather than letters (*e.g.* 'A' for addition & 'S' for subtraction; Knops, Thirion, et al., 2009). This likely minimized operation errors. Because a previous study (Katz & Knops, 2014) indicated greater variance for non-symbolic than symbolic trials, we decided to calculate each subject's mean separately for symbolic and non-symbolic notations and to exclude responses beyond mean plus/minus 3SD. This excluded 100 calculation trials (1% of calculation responses, all symbolic). Additionally, in 116 calculation trials (95 symbolic) and 7 catch trials (6 symbolic), no response was made within 6000ms (maximum response duration). The number of these timeout trials ranged from 0 to 37 trials per subject.

Attention task — In the Posner task, we first eliminated responses that were faster than 200ms, because these likely reflect premature responses. This eliminated 146 trials (7.3% of responses). The number of responses faster than 200ms per subject ranged from 0 to 37 responses (M = 8.6, SD = 10.4). We then eliminated responses where the RT was more than 3 SD from the subject's mean. This eliminated an additional 37 trials (2% of valid responses; 1-4 trials per subject, M = 2.2, SD = 1).

We computed validity effect (RT invalid minus RT valid), benefit (RT neutral minus RT valid) and cost (RT invalid minus RT neutral) as indices for orienting/selection and re-orienting, respectively.

Results

We first analyzed the distribution of responses to exclude the possibility that participants responded randomly. If participants responded non-randomly, then range and rank should have a significant effect on response choice (Katz & Knops, 2014). Indeed, we found that rank and range interacted for both multiplication and division in both symbolic and non-symbolic notation. We report these analyses in Appendix B.

Linear Increase of Response Value With Correct Value

Previous studies found that behavior was well described by Weber's law, suggesting a logarithmic compression of the underlying representation. This also appeared to be true in the present data (see Figure A.2). Transforming data log-scale also better meets the prerequisites of ANOVA, stipulating fixed variance (Katz & Knops, 2014). There was a linear increase of response value with correct value in both formats and operations (Table 1).

Table 1

Linear Increase of Response Value With Correct Value

	Linear scale			Log scale		
	t	Slope	95% CI	*t*	Slope	95% CI
Multiplication						
Symbolic	123.1**	0.96	0.94, 0.97	169.0**	0.98	0.97, 1.00
Non-symbolic	40.7**	1.24	1.18, 1.30	50.4**	1.05	1.01, 1.09
Division						
Symbolic	129.1**	0.96	0.95, 0.97	148.0**	0.98	0.97, 0.99
Non-symbolic	37.0**	0.95	0.90, 1.00	47.9**	0.92	0.89, 0.96

**Bonferroni corrected for multiple comparisons, $p < .013$.

Operational Momentum Effect

To investigate operational momentum, we entered the response bias, defined as the difference between the log chosen and the log correct values, into an ANOVA comprising the factors notation (symbolic, non-symbolic) and operation (multiplication, division). There was a significant interaction between operation and notation on response bias ($F(1,16) = 6.80$, $p = .018$). Since this makes interpretation of putative main effects difficult we no longer followed up on them. Therefore, simple main effects analysis was used to test the effect of operation, separately for symbolic and non-symbolic notations. For symbolic problems, operation did not have a significant effect on response bias (F < 1; Figure 3). However, symbolic multiplication (M = -0.007, 95% CI [-0.012, -0.002]), but not division (M = -0.006, 95% CI [-0.014, 0.001]), was significantly underestimated (i.e. mean response bias < 0; Figure 3).

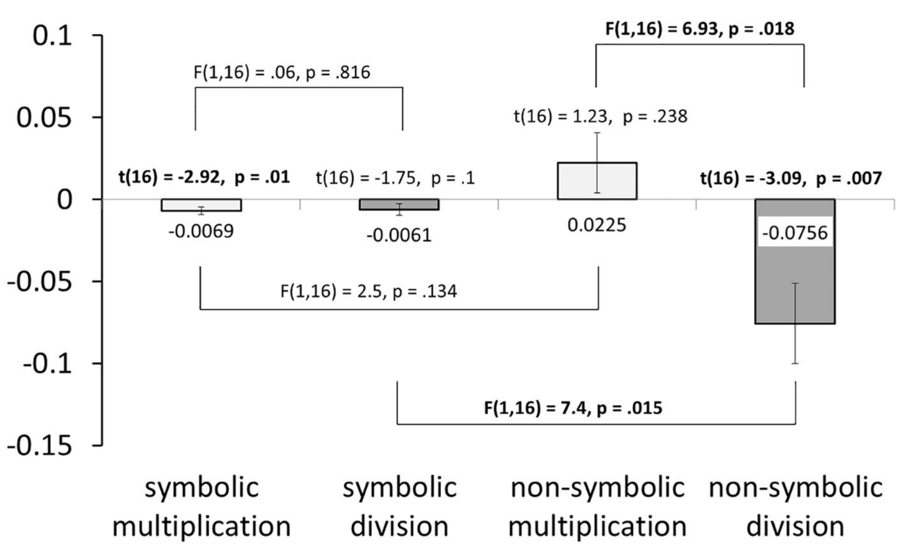

Figure 3. Response bias in symbolic and non-symbolic calculation. F-values represent simple main effects of operation (upper half) and notation (lower half). T-values represent one sample t-tests against a test value of zero. For symbolic problems (left), operation did not have a significant effect on response bias. The response bias for symbolic division (dark grey) was not significantly different than multiplication (light grey). However, symbolic multiplication problems were significantly underestimated. For non-symbolic problems (right), there was a significant effect of operation on response bias. Non-symbolic division (dark grey) problems were underestimated relative to multiplication (light grey). Only division, which was underestimated, showed a response bias significantly different from zero.

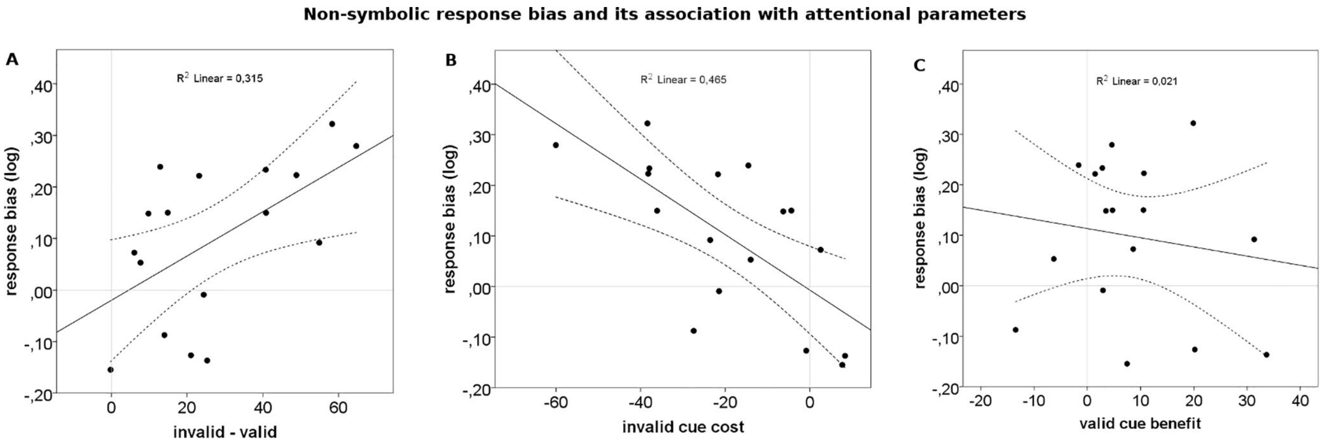

Figure 4. OM bias (log-scale) as a function of attentional parameters (A) validity effect (invalid – valid), (B) cost (neutral – invalid) and (C) benefit (neutral – valid) from the Posner task. Each data point corresponds to one subject.

For non-symbolic problems, operation had a significant effect on response bias ($F(1, 16) = 6.93$, $p < .05$, Bonferroni corrected; Figure 3). The mean response bias was positive for multiplication ($M = .023$, 95% CI [-0.016, 0.062]) and negative for division ($M = -0.076$, 95% CI [-0.127, -0.024]); difference = 0.098, 95% CI [0.019, 0.177]. Whereas we observed a significant underestimation in non-symbolic division ($t(16) = -3.09$, $p = .007$) we found no significant response bias in non-symbolic multiplication ($t(16) = 1.23$, $p = .238$; Figure 3).

Notation did not have a significant effect on response bias for multiplication ($F(1,16) = 2.50$, $p = .134$), but did for division ($F(1,16) = 7.40$, $p = .015$). The response bias was more pronounced (more negative) for non-symbolic than symbolic division ($t(16) = -2.72$, $p = .015$).

In sum, we replicated the results from Katz and Knops (2014), showing a significant OM-effect for non-symbolic multiplication and division but no OM-effect for symbolic operations, despite having encouraged approximate calculation by omitting the correct response from the symbolic response alternatives.

Can Attentional Orienting and Reorienting Predict Response Bias?

It has been put forward that the OM effect reflects the consequences of an attention-induced spatial displacement along the mental number line during the process of approximating the outcome of an arithmetic problem (Knops, Thirion, et al., 2009; Knops, Viarouge, & Dehaene, 2009; Knops, Zitzmann, & McCrink, 2013). Here, we tested the straight-forward hypothesis of an association between OM and spatial attention as measured in the Posner paradigm.

We first tested whether the relative OM bias, defined as the difference between operation-specific OM bias ((log correct minus log chosen Multiplication) minus (log correct minus log chosen Division)) correlated with the validity effect (valid cue minus invalid cue).

The validity effect was consistently larger than zero (i.e., faster response for valid than invalid) for all participants (M 27.5 ms; $t(16) = 5.64$, $p < .001$). To determine whether the validity effect could predict the difference in response bias between multiplication and division, each subject's validity effect was used as a predictor for the mean log-scale relative response bias (multiplication response bias minus division response bias) in a linear regression model. We restricted our analyses to the non-symbolic notation since symbolic OM bias was not significant. The advantage of a valid cue compared to an invalid cue significantly predicted non-symbolic relative response bias (F(1,16) = 6.90, p = .019. It accounted for 31.5% of the variability in response bias and had a large effect size ($\bar{R}^2$ = .269).

To further examine the attentional mechanisms potentially driving the OM effect in non-symbolic notation, we separately examined the effect of attentional orienting (benefit of valid cue compared to neutral cue; M = 8.3 ms, t(16) = 2.79, p = .013) and reorienting (cost of invalid cue relative to neutral cue; M = -19.2 ms, t(16) = -4.13, p = .001)) on response bias using linear regression. The adjusted R^2 value ($\bar{R}^2$) was used to determine effect sizes using the cutoffs: small = .01 or 1%, medium = .1 or 10%, and large = .25 or 25% (Vacha-Haase & Thompson, 2004). Invalid cue cost significantly predicted relative response bias in non-symbolic problems; F(1,16) = 13.02, p = .002, $\bar{R}^2$ = .464. Valid cue benefit could not predict response bias difference in non-symbolic problems; F(1,16) = 0.23, p = .638, $\bar{R}^2$ = -.044. Results are shown in Figure 4. The correlation between cost and response bias was significantly more negative than the correlation between benefit and response bias (Hotelling's t(13) = 2.15; Steiger's Z = 1.87, p < .05 (one-tailed)).

All correlations remained by-and-large unchanged after partialing out age as a potential confound (r(validity effect, response bias) = .613, p = .012; r(benefit, response bias) = -.088, p = .745; r(cost, response bias) = -.704, p = .002).

These results suggest that in non-symbolic problems, attentional shifts, most likely re-orienting but not orienting, largely account for the difference in response bias between multiplication and division.

Discussion

In this study we examined two questions. First, we explored whether the previously reported absence of an OM effect in symbolic multiplication and division (Katz & Knops, 2014) may have been due to the presentation of the correct outcome among the response alternatives which may have triggered verbally mediated fact retrieval and hence lowered the impact of visuo-spatial processes. In the present study, we found no OM effect in symbolic notation even though we encouraged approximate calculation by presenting exclusively incorrect symbolic response alternatives. In contrast, a significant OM effect was observed for non-symbolic notation, replicating Katz and Knops (2014). Second, we explored the underlying mechanisms of the OM effect by testing the association of the OM effect with visuo-spatial attention as measured by a Posner paradigm. We found that re-orienting attention after the presentation of an invalid cue to the location of the target significantly correlated with the extent to which participants over- and underestimated the outcomes of non-symbolic multiplication and division problems.

The absence of a significant OM in symbolic multiplication and division in the current study is in line with previous results and implies that symbolic multiplication and division strongly rely on verbally mediated fact retrieval which is less prone to cognitive biases such as the OM. While a recent study described how the compression of the mental number line biases arithmetic fact retrieval (Didino, Knops, Vespignani, & Kornpetpanee, 2015), these spatial biases may be too subtle for the current paradigm.

We demonstrated a significant correlation between non-symbolic OM and measure of reorienting attention after invalid cues in the Posner paradigm. No correlation was observed between OM and orienting attention after valid cueing. This is somewhat unexpected since the attention account of OM holds that attentional shifts propel participants too far along the mental number line. Hence, the benefit from a valid cue should correlate positively with OM. Yet, we did not observe any correlation between OM and cue benefit, a measure of attentional orienting. This might be due to a reduced variability in participants' performance in our Posner task which had slightly longer SOAs compared to the classical Posner paradigm. Variability may have been particularly reduced for the benefit measures as compared to cost measures, which, all else being equal, lowers correlation coefficients.

Reorienting is not a unitary process but has traditionally been subdivided into disengaging, shifting and reengaging attention to the new location (Posner & Petersen, 1990). In support of disengagement as a separate attentional mechanisms involved in reorienting, recent ERP studies revealed circumscribed posterior components linked with disengagement in the absence of attentional shifts (P4pc, Toffanin, de Jong, & Johnson, 2011), and separate from attentional selection (reversed N2pc, Eimer & Kiss, 2008). This more complex process may be subject to greater amount of variability across participants which, in turn, allows for higher correlation coefficients. Since attentional shifts and engaging are involved in both orienting and reorienting, the absence of a correlation between orienting and OM may as well imply that those processes that are unique to reorienting are at the heart of this association, namely disengaging. Future studies may use EEG to investigate this hypothesis.

Another question that arises from the current findings concerns the direction of the observed correlation. Why is the correlation between OM and reorienting positive, meaning that people with a large OM effect, i.e. larger deviations from the correct outcome exhibit larger costs for invalid cue both compared to neutral cues and valid cues? According to the above theoretical accounts of the OM we can break this question down into four aspects. Why would attentional reorienting correlate with (a) a heuristic according to which multiplication leads to larger outcomes and division to smaller outcomes, (b) flawed decompression, (c) competing spatial biases by the operands, the results or the outcome, or (d) attentional shifts along the mental number line?

According to the first account, no correlation with attentional measures would have been predicted. If any, there would have been a prediction that the expectation of larger or smaller outcomes creates attentional shifts to the left or right which might generate a coarse approximation of the result that can be used to check if a given outcome is plausible or not. However, what renders this hypothesis rather unlikely to account for the given results is that no gradation beyond a coarse "more or less" expectation is predicted.

Similarly, according to the compression-decompression approach OM results from flawed compression-decompression mechanisms. No attentional mechanism is involved in this process. According to this account, OM would scale with the size of both operands and results. However, previous research found that OM increases with increasing outcome (Knops, Viarouge, et al., 2009) but no association with operand magnitude was observed. In pointing tasks, OM was strongest when the second operand was zero, clearly speaking against any association between second operand size and OM (Pinhas & Fischer, 2008). Although the compression of the MNL is thought to be logarithmic, it is also subject to interindividual variance. A recent study tested whether the degree of compression of the approximate number system can actually serve to predict OM (Knops et al., 2014). However, the combination of crucial approximate number system parameters such as compression (as measured by the amount of underestimation in non-symbolic estimation) and precision (as measured by the Weber fraction) in a psychophysical model was not successful in predicting OM. While overall biases in addition and subtraction involving non-symbolic quantities were well predicted by the interindividual variability of the parameters describing the approximate number system in the model, the operation-specific OM was not. Finally, it is hard to see how this would conceptually relate to the costs of reorienting attention to an invalidly cued target. Further evidence against a heuristic-based account comes from the study of Magethis and colleagues (Marghetis et al., 2014) who found systematic biases in mouse pointer trajectories when participants indicated the correct outcome for addition and subtraction problems. Exploiting the fact that on-task tracking of mouse trajectories provides a time-resolved window on cognition, Margethis and colleagues found that the time course of this spatial deviation sequentially reflected the serial impact of first operand, operator and second operand. The authors conclude that neither a heuristic-based nor a compression account would predict this pattern of results, which is in line with a spatial account of the OM (Marghetis et al., 2014). Together, this suggests that the OM is most likely not fully accounted for by flawed compression-decompression mechanisms and suggests the origin in parameters outside the ANS.

With respect to the spatial accounts of the OM, a clear prediction comes from the attentional shift hypothesis which predicts a clear association between attentional parameters and OM. The current results partially confirmed this by the correlation between OM and reorienting, providing further evidence for a role of spatial attention during approximate arithmetic and, more specifically, for the idea that the OM results from attentional mechanisms. Larger reorienting costs may reflect highly efficient orienting mechanisms that need to be overruled after invalid cueing. These results are also in line with the finding that symbolic addition problems are solved faster when the second operand is presented on the right compared to left-sided presentation (Mathieu, Gourjon, Couderc, Thevenot, & Prado, 2016). For subtraction an analog advantage for left-sided operands was observed. Crucially, these authors also failed to find a benefit for lateralized operands in multiplication problems, highlighting the different cognitive processes that contribute to multiplication on the one side and addition or subtraction on the other. Recent results corroborate a tight link between the ocular movement system and OM (Klein et al., 2014). Relative to the first fixation, participants subsequently moved their eyes to the right for additions and to the left for subtractions, paralleling previous fMRI results (Knops, Thirion, et al., 2009). This dynamic process of adjusting fixation during the course of arithmetic processing may in part have been specific to the task which required indicating the outcome by pointing to the respective location on a number line. It is also conceivable, however, that approximate calculation is mediated by the dynamic updating within a coordinate transformation system in parietal cortex.

The spatial competition account, in contrast, does not predict any correlation between attentional shifts and OM. According to this account OM is largest when competition between spatial positions of operands and the results on the MNL are minimal. The strong OM in zero problems where only the first operand and the operational sign induce a spatial bias is explained by the absence of a spatial bias induced by zero (Pinhas & Fischer, 2008; Pinhas et al., 2015). Zero is either not represented on the MNL or triggers rule-based procedures. While we cannot test this prediction in the current experiment, we may interpret reorienting in terms of spatial competition between a cued position and the appearance of a target at an uncued position. The spatial competition account would predict that less competition is associated with larger OM. However, we observe the opposite pattern of results in the current study where larger amount of spatial competition is associated with larger OM. Hence, under the premise of interpreting the reorienting effect in terms of spatial competition our results provide evidence against this account. It should be noted, however, that the spatial competition account was initially proposed in the context of addition and subtraction. It is unclear whether it would also hold for multiplication and division where the split between operands and results is much larger.

Limitations

As this study provides only correlational evidence, no causal inference can be drawn. Further experimental work is required to elucidate the neurocognitive mechanisms underlying the OM effect. The absence of significant correlations between operational momentum bias and attentional measures may in part be due to the long SOAs in our version of the Posner task. Future studies could increase variability by using shorter SOAs, which might favor finding higher correlations. The fact that we related accuracy-based operational momentum bias with speed–based measures of the attentional capacities might further have reduced our statistical power. The limited sample size may raise concerns about (a) stability and reliability of the data and (b) to what extent the observed correlations were due to the increased variability in our sample. We checked whether our results were driven by some outliers by separately excluding all possible combinations of 1, 2 or 3 participants from the sample. We found single participants to have only a minor impact on the correlation pattern. Even when excluding the two participants with the most extreme value pairings, correlations by and large remained significant or marginally significant. To protect against the possibility that age was a confounding variable that drives our results, we recalculated the main findings after partialing out age and found the major findings of the study unchanged. Finally, while one may be concerned about the role of counting in computing the approximate results, we would reason that counting does play a major role in explaining our results. These additional analyses are can be found in Appendix C. Albeit OM and attention appear to be functional related on a conceptual level, future studies could use more comparable parameters for measuring these concepts.

Conclusions

To sum up, we failed to observe an OM effect in symbolic multiplication and division. This is in line with previous findings (Katz & Knops, 2014; Mathieu et al., 2016) and suggests that verbally mediated retrieval of arithmetic facts from long-term memory is less prone to spatial biases, yet not immune (Cavdaroglu & Knops, 2016; Didino et al., 2015). In contrast, we found a significant OM effect in non-symbolic notation, reflecting that participants systematically overestimated results of multiplication problems while underestimating results of division problems. The non-symbolic OM effect correlated with attentional parameters measured in a Posner paradigm. By differentially analyzing benefit and cost measures we found this correlation to mainly results from the costs of reorienting attention after invalid cues. While the exact mechanisms driving this correlation remain elusive, these results provide further evidence for the attentional shift hypothesis.

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Competing Interests

The authors have declared that no competing interests exist.

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Appendices

Appendix A: Arithmetic Problems and Response Alternatives

The problems we used were identical to the ones in Katz and Knops (2014). The responses for symbolic and non-symbolic problems were 0.5 × correct outcome < correct outcome < 2 × correct outcome and 0.33 × correct outcome < correct outcome < 3 × correct outcome, respectively. Eleven bins within the range of the response alternatives were created to form of a geometric series. The symbolic problems and the response range can be found in Table A.1.

Table A.1

Arithmetic Problems (Multiplication and Division), Including Response Alternatives

Operands		Response alternatives										
1st	2nd	1	2	3	4	5	6	7	8	9	10	11
Multiplication												
4	3	6	7	8	9	10	12	14	16	18	21	24
6	3	9	10	12	14	16	18	21	24	27	31	36
6	4	12	14	16	18	21	24	28	32	36	42	48
7	3	11	13	14	16	19	21	25	28	32	37	42
7	4	14	16	18	21	24	28	32	37	42	49	56
7	6	21	24	28	32	37	42	48	55	64	73	84
8	3	12	14	16	18	21	24	28	32	36	42	48
8	4	16	18	21	24	28	32	37	42	49	56	64
8	6	24	28	32	36	42	48	55	63	73	84	96
9	3	14	16	18	21	24	27	31	36	41	47	54
9	4	18	21	24	27	31	36	41	48	55	63	72
9	6	27	31	36	41	47	54	62	71	82	94	108
12	3	18	21	24	27	31	36	41	48	55	63	72
12	4	24	28	32	36	42	48	55	63	73	84	96
13	4	26	30	34	39	45	52	60	69	79	91	104
13	6	39	45	51	59	68	78	90	103	118	136	156
14	3	21	24	28	32	37	42	48	55	64	73	84
14	6	42	48	55	64	73	84	96	111	127	146	168
16	3	24	28	32	36	42	48	55	63	73	84	96
16	4	32	37	42	49	56	64	74	84	97	111	128
17	3	26	30	34	39	45	51	59	68	78	89	102
17	4	34	39	45	52	59	68	78	90	103	118	136
19	3	29	33	38	44	50	57	66	76	87	99	114
19	4	38	44	50	58	66	76	87	100	115	132	152
Division												
36	2	9	10	12	14	16	18	21	24	27	31	36
48	2	12	14	16	18	21	24	28	32	36	42	48
48	4	6	7	8	9	10	12	14	16	18	21	24
54	2	14	16	18	20	24	27	31	36	41	47	54
63	3	11	12	14	16	18	21	24	28	32	37	42
96	2	24	28	32	36	42	48	55	63	73	84	96
96	4	12	14	16	18	21	24	28	32	36	42	48
108	3	18	21	24	27	31	36	41	48	55	63	72
112	4	14	16	18	21	24	28	32	37	42	49	56
126	3	21	24	28	32	37	42	48	55	64	73	84

Operands		Response alternatives										
1^{st}	**2^{nd}**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
128	2	32	37	42	49	56	64	74	84	97	111	128
128	4	16	18	21	24	28	32	37	42	49	56	64
136	2	34	39	45	52	59	68	78	90	103	118	136
144	3	24	28	32	36	42	48	55	63	73	84	96
144	4	18	21	24	27	31	36	41	48	55	63	72
152	2	38	44	50	58	66	76	87	100	115	132	152
153	3	26	29	34	39	44	51	59	67	77	89	102
156	2	26	30	34	39	45	52	60	69	79	91	104
156	3	26	30	34	39	45	52	60	69	79	91	104
162	3	27	31	36	41	47	54	62	71	82	94	108
168	2	42	48	55	64	73	84	96	111	127	146	168
168	4	21	24	28	32	37	42	48	55	64	73	84
171	3	29	33	38	43	50	57	65	75	86	99	114
192	4	24	28	32	36	42	48	55	63	73	84	96

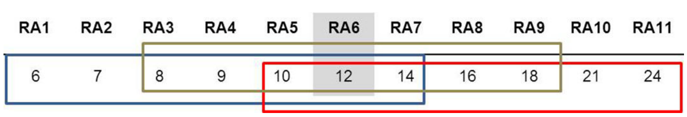

Figure A.1. Exemplary depiction of how we varied the ordinal rank of the correct outcome (gray column) within the low (blue), middle (brown), and high (red) range of response alternatives (RA 1 to RA 11). Example depicts the outcomes for the problem 3 × 4 in the symbolic notation.

Appendix B: Analysis of Interaction Between Range and Rank to Ensure Participants Were not Responding Randomly

Symbolic Multiplication

There was a significant interaction between range and rank ($F(12,192) = 143.315$, $p < .001$, $\varepsilon = .143$). Simple main effects analysis revealed a significant effect of rank on response percentage for all response ranges (low, medium, high). For all response ranges, post-hoc pairwise comparisons confirmed that the correct choice was selected significantly more often than all other choices (Table A.2).

Table A.2

Simple Main Effect of Rank on Response Percentage

				Sphericity[a]	
Range	$F(6,96)$	p	**Partial η^2**	χ^2	ε
Symbolic multiplication					
Low, 5th correct**	168.847	<.001	.913	157.0	.230
Med., 4th correct**	131.614	<.001	.892	132.2	.243
High, 3rd correct**	207.360	<.001	.928	183.4	.275
Non-symbolic multiplication					
Low, 5th correct**	7.209	.001	.311	39.5	.474
Med., 4th correct	2.628	.058	.141	35.1	.522
High, 3rd correct*	3.084	.038	.162	47.3	.484
Symbolic division					
Low, 5th correct**	109.958	<.001	.873	118.6	.266
Med., 4th correct	142.816	<.001	.889	118.6	.283
High, 3rd correct	116.792	<.001	.880	127.7	.288
Non-symbolic division					
Low, 5th correct**	3.043	.046	.160	59.4	.433
Med., 4th correct*	4.916	.012	.235	75.8	.350
High, 3rd correct**	14.402	<.001	.474	83.0	.355

[a]Greenhouse-Geisser corrected for violations of sphericity as measured by Mauchley's Test of Sphericity.

*Significant simple main effect only.

**At least 1 significant post-hoc pairwise comparison (Bonferroni-corrected $p < .05$).

Symbolic Division

There was a significant interaction between range and rank ($F(12,192) = 121.256$, $p < .001$, $\varepsilon = .133$). Similar to symbolic multiplication, simple main effects analysis revealed a significant effect of rank on response percentage for all response ranges and post-hoc pairwise comparisons confirmed that the correct choice was selected significantly more often than all other choices (Table A.3).

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Table A.3

Estimated Marginal Means, Post-Hoc Pairwise Comparisons

	1	2	3	4	5	6	7
Symbolic multiplication							
L	$0.1^{3,4,5,6,7}$	$0.1^{3,4,5,6,7}$	$0.3^{1,2,4,5}$	$1.2^{1,2,3,5,7}$	$5.6^{1,2,3,4,6,7}$	$1.0^{1,2,5}$	$0.4^{1,2,4,5}$
M	$0.2^{2,3,4}$	$0.4^{3,4,5,7}$	$1.2^{1,2,4,6,7}$	$5.2^{1,2,3,5,6,7}$	$1.1^{1,2,4,6,7}$	$0.2^{3,4,5}$	$0.1^{2,3,4,5}$
H	$0.4^{2,3,4,6,7}$	$1.3^{1,3,5,6,7}$	$5.4^{1,2,4,5,6,7}$	$1.0^{1,3,5,6,7}$	$0.1^{2,3,4}$	$0.0^{1,2,3,4}$	$0.0^{1,2,3,4}$
Symbolic division							
L	$0.1^{4,5,6}$	$0.2^{4,5,6}$	$0.3^{4,5,6}$	$1.2^{1,2,3,5,7}$	$5.0^{1,2,3,4,6,7}$	$1.1^{1,2,3,5,7}$	$0.4^{4,5,6}$
M	$0.3^{3,4,5}$	$0.2^{3,4,5}$	$1.2^{1,2,4,6,7}$	$5.0^{1,2,3,5,6,7}$	$1.0^{1,2,4,6,7}$	$0.3^{3,4,5,7}$	$0.1^{3,4,5,6}$
H	$0.4^{2,3}$	$1.4^{1,3,5,6,7}$	$5.0^{1,2,4,5,6,7}$	$1.0^{3,5,6,7}$	$0.3^{2,3,4}$	$0.1^{2,3,4}$	$0.0^{2,3,4}$
Non-symbolic multiplication							
L	1.0^{5}	$1.0^{4,5,6}$	$1.0^{5,6}$	1.2^{2}	$1.5^{1,2,3}$	$1.8^{2,3}$	1.6
M	1.0	1.0	1.1	1.6	1.4	1.5	1.0
H	1.2	1.3	1.6	1.3	1.4	1.1	1.0
Non-symbolic division							
L	1.0	1.4	1.5^{6}	1.4^{6}	1.4^{6}	$0.8^{3,4,5}$	1.0
M	1.6	1.7	1.5	1.3	1.0	0.8	0.6
H	$2.4^{6,7}$	$2.1^{3,4,5,6,7}$	$1.3^{2,6,7}$	$1.0^{2,6}$	$0.8^{2,6}$	$0.4^{1,2,3,4,5}$	$0.5^{1,2,3}$

Note. L = Low-5th correct. M = Medium-4th correct. H = High-3rd correct.

^{1,2,3,4,5,6,7} Significantly different (Bonferroni-corrected) from rank ^{n}

Non-Symbolic Multiplication

There was a significant interaction between range and rank ($F(12,192) = 6.940$, $p < .001$, ε = .566). Simple main effects analysis revealed a significant effect of rank on response percentage for trials where the low high response range was presented, but not for the medium range. When the medium and high ranges of response choices were presented, there were no significant differences in response percentages between any of the seven response choices (Table A.2). Therefore, random responding cannot be excluded in medium and high range non-symbolic multiplication trials. In the low range, the 6th choice (too large) was selected significantly more than the 2nd or 3rd choice, the 5th choice (correct choice) was selected significantly more often that the 1st, 2nd or 3rd and the 4th choice significantly more than the 2nd choice (Table A.3). Thus, the percentage of responses at the 4th, 5th and 6th choice were not significantly different for low range trials.

Non-Symbolic Division

There was a significant interaction between range and rank ($F(12,192) = 10.337$, $p < .001$, ε = .414). Simple main effects analysis revealed a significant effect of rank on response percentage for all response ranges. However, after correction for multiple comparisons, post-hoc pairwise comparisons revealed significant differences between the seven response choices only for low and high range trials (Table A.2). When the medium range of response choices was presented, there were no significant differences in response percentages between any of the seven response choices (Table A.2). This indicates that although random responding cannot be excluded in medium range non-symbolic division trials, it can be in both low and high range trials. In low range trials, the 3rd (too small), 4th (too small) and 5th (correct) choices were selected significantly more than the 6th choice (too large), but there was no significant difference between the 1st through 5th choice (Table A.3). In high range trials, the smallest choices (1st, 2nd & correct, 3rd) were selected significantly more often than the two largest choices (6th & 7th) and the 6th was selected less often than all of the smaller choices (1st-5th). The 2nd choice was selected more often than all of the larger choices (3rd-7th), including the correct choice (Table A.2). This indicates significant pattern of choosing one smaller (2nd choice) than the correct choice when the 3rd choice was correct (high range), a possibly random response pattern when the middle choice (4th) was correct (medium range) and a non-random response pattern driven by a decreased likelihood of choosing the 6th choice than the 3rd, 4th and 5th when the 5th was correct (low range).

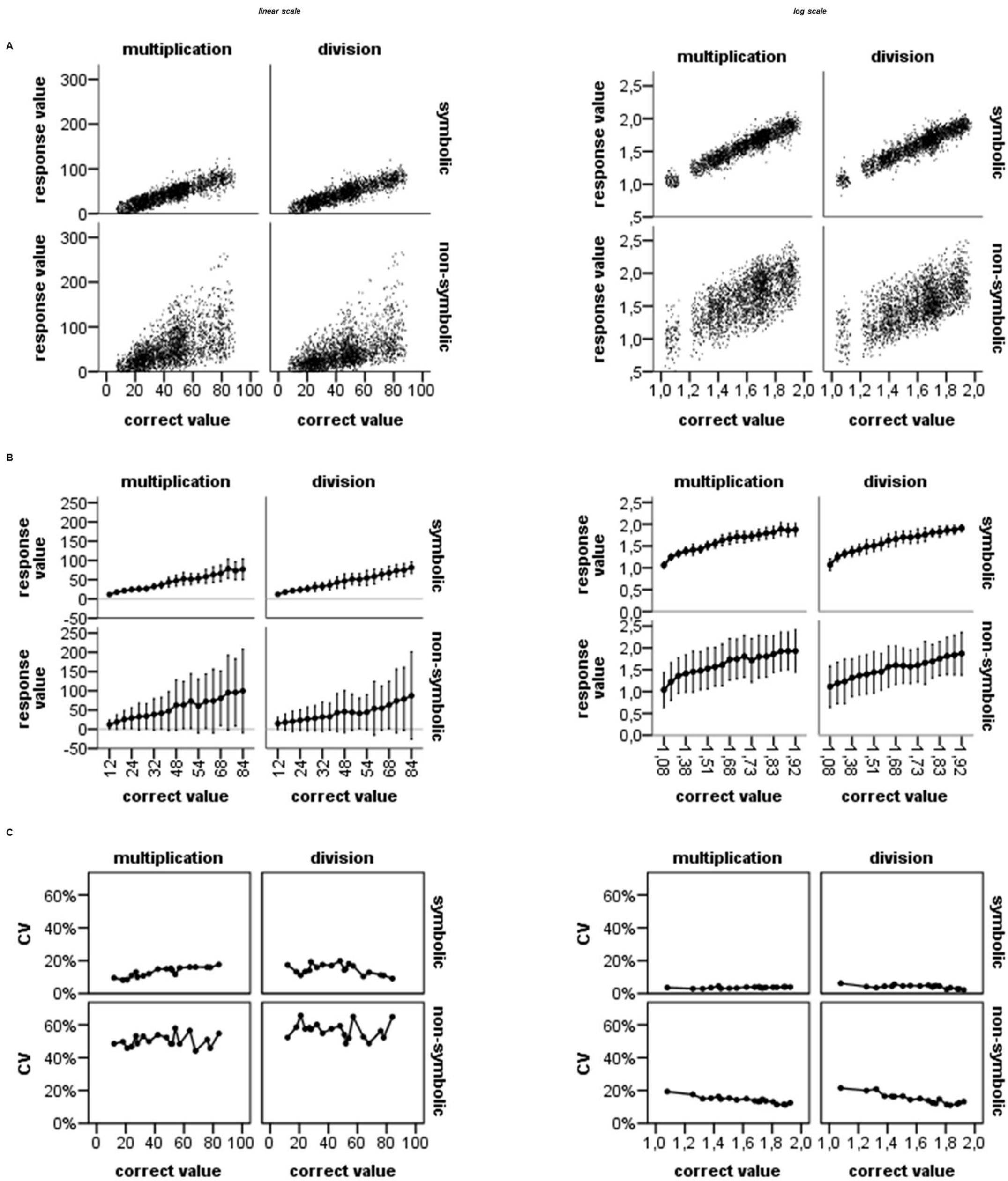

Figure A.2. Response value as a function of correct value on linear and log-scale non-aggregated data. (A) The non-aggregated response value was plotted as a function of correct value on the linear (left) and log (right) scale. Each point represents one case and number of cases is indicated by dot density (*i.e.* darker color). (B) As predicted by Weber's law, linear response value and variability (error bars represent ± 2 SD) (left) increased as a function of correct value. This could clearly be seen in non-symbolic problems. When the $\log_{10}$ of the response and correct value was used, response value increased as a function of correct value, but variability remained constant. (C) The dispersion of the response choices, measured by the mean-centered coefficient of variation (CV), was more constant when the log-scale data was used (right).

Appendix C: Analyses Concerning Reliability and Stability of the Observed Results

In the following we report the results of additional analyses (1) to verify the stability and reliability of the data, as well as the (2) impact of age, a (3) Bayesian analysis of the reported null effects, and (4) why we think counting does not play a major role in the current experiment.

1. Are the Observed Correlations Due to Outliers?

We checked the effect of separately excluding all possible combinations of 1, 2 or 3 participants from the sample on our main findings, that is the correlation between

(1) the validity effect (RT valid – RT invalid) and the OM bias (OM_multiplication – OM_division)

Excluding 1 participant: When excluding the two participants (#13 and #16) marked in red in the Figure A.3, correlation was *r* = .485 and significance drops to a marginal significance of *p* = .0569. These two participants were aged 25 and 29, hence not in the extreme age range of our sample (19-74).

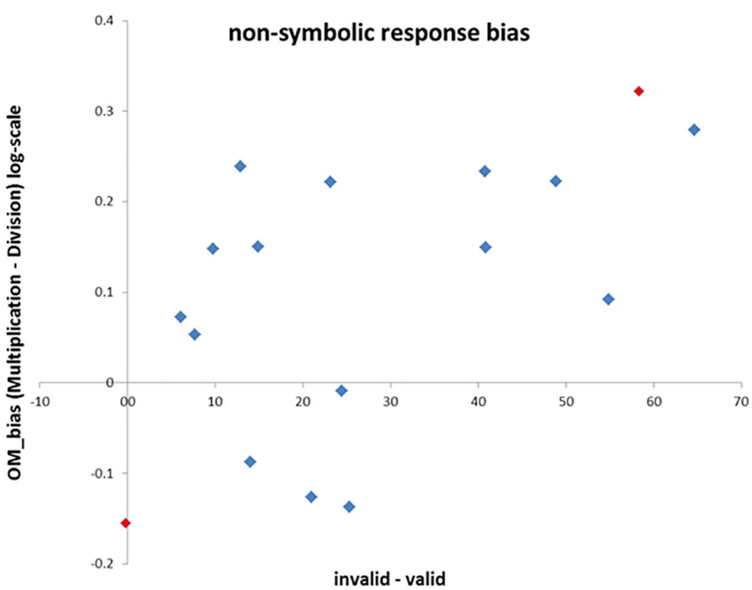

Figure A.3. Scatter plot of the validity effect against the OM bias. Each dot represents one participant.

When excluding more than one participant, 32 out of 136 combinations of two participants (23.5%), or 229 out of 680 combinations of three participants were not significant (33.7%).

(2) the benefit and the OM bias

Excluding one participant did not change the fundamental pattern of results. All correlations remained significant (p > .05).

When excluding more than one participant, 0 out of 136 combinations of two participants (0%), or 11 out of 680 combinations of three participants were not significant (1.62%).

(3) the cost and the OM bias

Only when excluding one participant (#17) the correlation (*r* = -. 489) was marginally significant only (*p* = .055).

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When excluding more than one participant, 20 out of 136 combinations of two participants (14.7%), or 183 out of 680 combinations of three participants were not significant (26.9%).

2. Are Our Results Due to Some Confounding Impact of Age?

Partialing out age did not significantly impact the correlation between OM bias and validity effect (r_{age}(bias, validity effect) = .613, (p = .0115) or the correlation between cost and OM bias (r_{age}(bias, cost) = .70, p = .002), both of which were significant before partialing out the influence of age, too. The partial correlation between benefit and OM bias corrected for age (r_{age}(bias, benefit) = -.09, p = .75) was not significant, thus remained unchanged, too.

Together with the above results we think that these results imply rather stable data even given the small sample size. Age was not a major factor driving our results. On the contrary, our convenience sample increases the generalizability of the data to the population since we do not – as the majority of studies in the field of experimental psychology – artificially restrict our sample to university students in their early twenties. Further, it should be mentioned that we chose a rather conservative procedure when excluding one participant before engaging in our analyses based on the participant's low performance.

3. Bayesian Analysis of the Correlation Between Benefit and OM Bias.

For the non-significant correlation between benefit and OM bias, a Bayes factor BF10 = 0.214 indicates evidence in favor of the null hypothesis. It is 1/0.214 = 4.673 more likely that the data occurred under the H0 than the H1 (Wetzels & Wagenmakers, 2012).

4. Does Counting Play a Major Role in Explaining Our Results?

For the following reasons we consider counting as an unlikely strategy to account for the results. The majority of operands cannot be counted during two seconds. Assuming a counting rate of ~250 ms per item, participants may have counted up to ~8 items. This allowed counting only those problems where the sum of the operands would be around 8. This was the case for four problems only. What makes it even more unlikely that participants counted is that the results screen contained seven dot patterns, clearly exceeding the time limit to count all dots within only 6 seconds. Mean reaction time for non-symbolic problems was 2.595 second (*SD* = 1.129), showing that the vast majority of responses was provided within less than 3.7 seconds (*M* + 1 *SD*). Even if participants were able to count one of the operands (e.g. where it was 2, 3 or 4), this means that participants had to divide or multiply an approximate second quantity by/with an exact value, again leading to approximate values. Last, we think it is very unlikely that participants engaged in tedious counting routine over a period of approximately 1 hour.

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