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Subtraction by Addition Strategy Use in Children of Varying Mathematical Achievement Level: A Choice/No-Choice Study

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Abstract

We investigated the use of the subtraction by addition strategy, an important mental calculation strategy in children with different levels of mathematics achievement. In doing so we relied on Siegler's cognitive psychological model of strategy change (Lemaire & Siegler, 1995), which defines strategy competencies in terms of four parameters (strategy repertoire, distribution, efficiency and selection), and the choice/no-choice method (Siegler & Lemaire, 1997), which is essentially characterized by offering items in two types of conditions (choice and no-choice). Participants were 63 11-12-year-olds with varied mathematics achievement levels. They solved multi-digit subtraction problems in the number domain up to 1,000 in one choice condition (choice between direct subtraction or subtraction by addition on each item) and two no-choice conditions (obligatory use of either direct subtraction or subtraction by addition on all items). We distinguished between two types of subtraction problems: problems with a small versus large difference between minuend and subtrahend. Although mathematics instruction only focused on applying direct subtraction, most children reported using subtraction by addition in the choice condition. Subtraction by addition was applied frequently and efficiently, particularly on small-difference problems. Children flexibly fitted their strategy choices to both numerical item characteristics and individual strategy speed characteristics. There were no differences in strategy use between the different mathematical achievement groups. These findings add to our theoretical understanding of children's strategy acquisition and challenge current mathematics instruction practices that focus on direct subtraction for children of all levels of mathematics achievement.

Keywords: subtraction by addition, multi-digit subtraction, strategy flexibility, model of strategy change, choice/no-choice method, mathematical achievement level

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Cumulative evidence indicates that children and adults rely on a rich diversity of mental calculation strategies to solve elementary arithmetic tasks (e.g., Campbell, 1999, 2008; LeFevre, Bisanz, et al., 1996; LeFevre, Sadesky, & Bisanz, 1996). In the domain of multi-digit subtraction, adults have shown to apply two different types of mental calculation strategies: direct subtraction and subtraction by addition (Peters, De Smedt, Torbeyns, Ghesquière, & Verschaffel, 2010; Torbeyns, Ghesquière, & Verschaffel, 2009). *Direct subtraction* strategies are characterized by the use of a subtraction operation to solve a problem: The subtrahend is taken away from the minuend (e.g., solving 82 - 67 = __ via 82 - 60 - 7 = 22 - 7 = 15). By contrast, *subtraction by addition* strategies involve the use of the addition operation to answer the problem: The difference between the minuend and the subtrahend is computed by adding on from the subtrahend up to the minuend (e.g., answering 82 - 67 = __ via 67 + 3 = 70 and 70 + 12 = 82, so the answer is 3 + 12 or 15). Subtraction by addition is assumed to be highly efficient on subtraction problems with a small difference between minuend and subtrahend, such as 81 - 79, given the small number and relative ease of the steps in the solution process (Blöte, Klein, & Beishuizen, 2000; Peters et al., 2010; Selter, 2001; Torbeyns et al., 2009).

On the other hand, previous studies have revealed that children hardly report the subtraction by addition strategy on multi-digit subtraction problems (Blöte, Van der Burg, & Klein, 2001; De Smedt, Torbeyns, Stassens, Ghesquière, & Verschaffel, 2010; Heinze, Marschick, & Lipowsky, 2009; Selter, 2001; Selter, Prediger, Nührenbörger, & Hussmann, 2012; Torbeyns, De Smedt, Ghesquière, & Verschaffel, 2009a, 2009b; but see Peltenburg, van den Heuvel-Panhuizen, & Robitzsch, 2012 for an exception). One important difference between these adult and child studies involves the research methods used. Whereas the adult studies applied the choice/no-choice method (Siegler & Lemaire, 1997), the children's data were collected in an experimental setting wherein they simply were invited to solve the subtraction problems with the strategy of their choice. Taking into account this important methodological difference between the adult and child studies, the present study investigates the subtraction by addition strategy in children with the choice/no-choice method (Siegler & Lemaire, 1997). Compared to the previous studies with children, we additionally included a comparison of mathematically low to mathematically high achievers' use of the direct subtraction and subtraction by addition strategy in the domain of multi-digit subtraction up to 1,000.

Studying People's Strategy Competencies

The theoretical and methodological framework of R. S. Siegler (Lemaire & Siegler, 1995; Siegler & Lemaire, 1997) significantly contributed to the study of strategy choice and change processes (Siegler, 2003; Torbeyns, Arnaud, Lemaire, & Verschaffel, 2004; Verschaffel, Greer, & De Corte, 2007). In their *model of strategy change*, Lemaire and Siegler (1995) distinguish four parameters of strategy competencies: (1) strategy repertoire, defined as the different types of strategies people use to solve arithmetic tasks; (2) strategy distribution, referring to the frequency with which each of these strategies is applied; (3) strategy efficiency, defined as the accuracy and speed of strategy execution; and (4) strategy selection, referring to the flexibility of strategy choices. Lemaire and Siegler defined strategy flexibility based on individual strategy efficiency competencies: selecting the strategy that most quickly brings one to an accurate answer to the problem. According to their model, the development of strategy competencies involves changes in the four parameters. Changes in the strategy repertoire can occur from explicit teaching, but people can also invent new strategies based on increasing practice and (consequently) fluent mastery of available strategies, which allows them to devote attentional resources to looking for shortcuts and to developing alternatives (Shrager & Siegler, 1998).

Siegler and Lemaire (1997) have proposed the *choice/no-choice method* to empirically investigate people's strategy competencies. This method requires assessing participants in two different types of conditions: a *choice condition* where they are allowed to choose the strategy for each problem, and two or more *no-choice conditions* where they are forced to solve all problems with a given strategy. The no-choice conditions make it impossible to select strategies based on problem and/or individual characteristics, resulting in unbiased efficiency data for the strategies involved in the study. Moreover, the choice/no-choice method allows one to analyse the flexibility of participants' strategy choices based on their individual strategy efficiency competencies by comparing their strategy choices in the choice condition with their individual strategy efficiency data collected in the no-choice conditions.

Multi-Digit Subtraction Strategies

Since the end of the previous century, the acquisition of various strategies, which can be applied efficiently and flexibly on different types of mathematical tasks, has become a major goal of primary mathematics education worldwide (Baroody & Dowker, 2003; Kilpatrick, Swafford, & Findell, 2001; Verschaffel, Greer, & De Corte, 2007). According to the adherents of the worldwide reform movement, teaching for strategy variety and flexibility serves as a major stepping stone towards the insightful acquisition of these strategies and continued flexibility in children's mathematical thoughts and practices. Striving for strategy variety and flexibility is present, for instance, in reform-oriented mathematics curricula in the U.S. (e.g., National Council of Teachers of Mathematics, 2003; Parrish, 2014) and in various European countries (e.g., van Zanten & van den Heuvel-Panhuizen, in press; Wittmann & Müller, 2007), including Flanders (Belgium), where the present study was conducted (Vlaams Ministerie van Onderwijs en Vorming, 2010). This international striving for strategy variety and flexibility also applies to the domain of multi-digit subtraction. To solve multi-digit subtraction problems, children can apply diverse strategies. These strategies can be classified in two different but complementary ways (see Peltenburg et al., 2012; Torbeyns, Ghesquière, et al., 2009): on the basis of the manipulation of the numbers in the problem (the *number perspective*) or on the basis of the model of the operation being selected (the *operation perspective*).

The first classification, the *number perspective*, relates to the distinction among three types of strategies: decomposition, sequential and varying strategies (cf. Beishuizen, 1993; Blöte et al., 2000, 2001; Selter, 1998, 2001; Verschaffel et al., 2007). Decomposition strategies involve decomposing both minuend and subtrahend along their decimal structure (decomposing both integers into hundreds, tens, and units) and subtracting them separately (e.g., solving 628- 313 = __ via 600 - 300 = 300, 20 – 10 = 10, 8 – 3 = 5; so the answer is 300 + 10 + 5 = 315). Sequential strategies start the calculation process with the un-split minuend and then the hundreds, tens and units of the subtrahend are handled sequentially (e.g., solving 628 - 313 = __ via 628 - 300 = 328, 328 - 10 = 318, 318 - 3 = 315). Varying strategies involve the flexible adaptation of the numbers in the problem on the basis of one's understanding of number relations and/or the properties of the arithmetic operations. A well-known example of this latter type of strategies is the compensation strategy, which can be applied on problems with the subtrahend ending on the digit 8 or 9 (e.g., solving 628 - 399 = __ via 628 - (400 - 1) = 228 + 1 = 229).

The second classification of multi-digit subtraction strategies, the *operation perspective*, is based on the model of the operation that underlies the solution process. This classification distinguishes two major types of strategies: direct subtraction and subtraction by addition[i] As indicated by the names of the strategies and as outlined above, direct subtraction strategies are characterized by the straightforward use of the subtraction operation to solve subtraction problems: they involve the direct subtraction of the subtrahend from the minuend (e.g., solving 628 - 313 = __ via 628 - 300 = 328, 328 - 10 = 318, 318 - 3 = 315). By contrast, subtraction by addition is characterized by the use of the complementary addition operation to solve a given subtraction problem: subtraction by addition strategies require to determine how much should be added to the subtrahend to arrive at the minuend (e.g., solving 628 - 313 = __ via 313 + 87 = 400, 400 + 200 = 600, 600 + 28 = 628, so the answer is 87 + 200 + 28 or 315). Whereas the direct subtraction strategy fits nicely with the operational model of subtraction as “taking away”, subtraction by addition corresponds with the “bridging the difference” model of subtraction (Selter et al., 2012). As argued in Peltenburg et al. (2012), both direct subtraction and subtraction by addition strategies can by executed in different ways - by means of a decomposition, sequential or varying strategy. The present study focuses on the second kind of classification, namely children’s use of direct subtraction versus subtraction by addition strategies.

Previous Studies on Direct Subtraction Versus Subtraction by Addition Use

Following a choice design, with subtraction problems offered in only one choice condition, previous investigations on children’s direct subtraction versus subtraction by addition strategy use revealed that children hardly report using the latter strategy to solve multi-digit subtraction problems (Selter, 2001; Selter et al., 2012; Torbeyns, De Smedt, et al., 2009a, 2009b). Moreover, intervention studies indicate that even after explicit instruction in subtraction by addition, direct subtraction remains children’s preferred strategy in the domain of multi-digit subtraction (Blöte et al., 2000, 2001; De Smedt et al., 2010; Heinze et al., 2009; Selter, 2001; Selter et al., 2012; Torbeyns, De Smedt, et al., 2009a, 2009b). As stated above, these results sharply contrast with the results from adult studies that systematically revealed the frequent, efficient and flexible application of subtraction by addition on multi-digit subtraction problems (Verschaffel, Torbeyns, De Smedt, Peters, & Ghesquière, 2010).

Following a choice/no-choice design, Torbeyns, Ghesquière, et al. (2009) and Torbeyns, De Smedt, Peters, Ghesquière, and Verschaffel (2011) empirically addressed adults’ strategy use on multi-digit subtraction problems. In three subsequent studies, Torbeyns et al. (2009, 2011) offered adults different types of multi-digit subtraction problems with numbers up to 1,000 in one choice and two no-choice conditions. In the choice condition, participants had to choose between direct subtraction and subtraction by addition. In the two no-choice conditions, all subtraction problems had to be answered via direct subtraction and subtraction by addition, respectively. Torbeyns et al. constructed two types of subtraction problems based on the difference between minuend and subtrahend: subtraction problems with an extremely small difference (a three-digit minuend, a three-digit subtrahend and a two-digit difference; e.g., 713 - 695 = 18) and subtraction problems with an extremely large difference (a three-digit minuend, a two-digit subtrahend and a three-digit difference; e.g., 613 - 67 = 546). The verbal strategy reports from the choice condition revealed that the majority of the participants applied the subtraction by addition strategy at least once. Moreover, they used this strategy more frequently than the direct subtraction strategy. Surprisingly, in the no-choice conditions subtraction by addition was executed faster and more accurately than direct subtraction, not only on small-difference subtraction problems (for which subtraction by addition is assumed to be the most efficient) but also on large-difference subtraction problems (for which direct subtraction is expected to be the most efficient). Finally, participants flexibly fitted their strategy choices to both the numerical characteristics of the items and to their individual strategy performance characteristics. In the choice condition, small-difference subtraction problems were most frequently solved via subtraction by addition, and adults applied subtraction by addition more when it was the most efficient strategy for them.

The Present Study

Against the background of the above-mentioned findings in adults, we aimed to analyse children’s use of the direct subtraction versus subtraction by addition strategy relying on the same theoretical framework and research method. As outlined above, previous investigations on children's direct subtraction versus subtraction by addition strategy use (Blöte et al., 2001; De Smedt et al., 2010; Heinze et al., 2009; Peltenburg et al., 2012; Selter, 2001; Selter et al., 2012; Torbeyns, De Smedt, et al., 2009a, 2009b) are limited by an important methodological weakness. These studies followed a simple choice design, constraining the findings on strategy efficiency and flexibility. Because children were only tested in a choice condition, it is not clear whether their highly infrequent verbally reported use of subtraction by addition was due the absence of this strategy in their strategy repertoire or due to weak mastery of this strategy as compared to the well-trained direct subtraction strategy. To address this methodological weakness, we applied a choice/no-choice design and analysed children's subtraction by addition strategy use to solve multi-digit subtraction problems along the four parameters of Lemaire and Siegler's (1995) model of strategy change. We also investigated the use of subtraction by addition as a function of children's mathematical achievement level. In doing so, we aimed to increase both our theoretical understanding of multi-digit subtraction strategy acquisition in children with various levels of mathematical achievement and the potential educational relevance of our work.

Using the four parameters of the model of strategy change, we formulated the following research questions (RQ).

RQ1: Do primary school children verbally report subtraction by addition on subtractions up to 1,000 in the choice condition (= strategy repertoire)?

RQ2: How frequently do they apply subtraction by addition on subtractions up to 1,000 in the choice condition (= strategy distribution)?

RQ3: How accurately and quickly do they apply subtraction by addition compared to direct subtraction on subtractions up to 1,000 in the no-choice conditions, in general and on the different types of subtractions (= strategy efficiency)?

RQ4: Do they fit their strategy use to the numerical characteristics of the subtraction problems and/or to their individual strategy efficiency competencies (= strategy selection)?

RQ5: To what extent do the answers to these first four research questions differ between children of varying mathematical achievement levels?

Method

Participants

Participants were 68 sixth-graders from two primary schools in Flanders (Belgium) (M_{age} = 11y8m [*SD* = 4m]; 29 boys). We selected two primary schools located in two different provinces in Flanders. We invited all sixth-graders from these randomly selected schools to participate to the study. We chose sixth grade for two reasons. First, in contrast to younger learners, sixth-graders had sufficient mastery of multi-digit addition and subtraction to apply the untaught subtraction by addition strategy after only a brief demonstration. Second, in contrast to older learners, sixth-graders still frequently practice multi-digit subtraction during their mathematics lessons.

We analysed children's mathematics instruction history via scrutinized textbook analyses and structured teacher interviews. These revealed that all children had received instruction on multi-digit subtraction problems from 2nd grade on. In line with typical mathematics practices in Flanders, instruction focused on mental calculation through direct subtraction, and more specifically on the sequential direct subtraction strategy. After sufficient practice of this sequential direct subtraction strategy, children received instruction on other direct subtraction strategies, including varying strategies such as compensation. However, as is typically the case in Flanders (Torbeyns, De Smedt, et al., 2009a, 2009b), none of the children had received explicit instruction in the subtraction by addition strategy.

We assessed children's general mathematical achievement level with a standardized mathematical achievement test, addressing all domains of the mathematics curriculum in sixth grade (Deloof, 2005). Based on these test scores, children were divided into four mathematical achievement groups: (1) *low* achievers (Pc < 16), (2) *below-average* achievers (Pc 16-50), (3) *above-average* achievers (Pc 51-84), and (4) *high* achievers (Pc > 84). Low and high mathematics achievement were defined as performing 1 standard deviation below or above the population average (which in a normal distribution correspond to the 16th [low] and 84th [high] percentiles). As elaborated in the Procedures section, the data of five participants (two low achievers, two above-average achievers, one high achiever) had to be discarded. The final sample consisted of 63 participants (Table 1). These achievement groups differed significantly in their mean percentile scores on the mathematics achievement test (Deloof, 2005), $F(3, 59) = 298.7$, $p < .01$, $BF_{10} = 9.8 \times 10^{31}$.

Table 1

Number of Boys/Girls and Mathematics Achievement Percentile Score per Mathematical Achievement Group (n = 63)

	n		**Mathematics Achievement Percentile Score**	
Achievement Group	**Boys**	**Girls**	**Mean**	**Standard Deviation**
Low	2	6	8.25	5.20
Below-average	9	6	38.33	9.94
Above-average	9	9	66.94	9.10
High	8	14	93.59	5.21

Materials

Children were offered three parallel series of 13 multi-digit subtraction problems up to 1,000. In line with Torbeyns et al. (2011) we distinguished two types of subtraction problems (with five subtraction problems each): (1) *small-difference subtraction problems* (SD), defined as subtraction problems with a small difference between minuend and subtrahend: a three-digit minuend, a three-digit subtrahend and a two-digit difference (e.g., 614 - 596 = 18); (2) *large-difference subtraction problems* (LD), characterized by a large difference between minuend and subtrahend: a three-digit minuend, a two-digit subtrahend and a three-digit difference (e.g., 734 - 47 = 687). We included three buffer items consisting of a three-digit minuend, a three-digit subtrahend and a three-digit difference that was about half of the minuend (e.g., 634 - 278 = 356). These buffer items were not assumed to specifically stimulate either direct subtraction or subtraction by addition strategy use.

All subtraction problems required crossing-over both tens and units. To dissuade children from using computation strategies that differ from the requested direct subtraction and subtraction by addition strategy (see below, conditions), we did not include subtraction problems with unit values of 0, 5 or 9. To match the difficulty of the three series of subtraction problems, we equated the mean sizes of the minuends and subtrahends as well as the mean sizes of the differences across the series.

In each of the three series subtraction problems were ordered based on three criteria: (1) unpredictable order of the different types of problems; (2) no repetition of problem type on two consecutive trials; (3) the answer to each subtraction problem could not easily be derived from the previous one. We created two orders per series, with order 2 being the reverse of order 1.

Conditions

Children were interviewed individually in one choice and two no-choice conditions. In the choice condition, they could choose between direct subtraction and subtraction by addition on each problem. In the experimental instructions, animations (a boy and a girl) were used to explain the strategies and steps that were allowed. To allow valid comparisons of children's direct subtraction versus subtraction by addition strategy competencies, children had to apply one specific sequential variant of direct subtraction and subtraction by addition (and were thus not allowed to use a rich variety of decomposition, sequential and varying direct subtraction versus subtraction by addition strategies). The solution steps were visible on the screen, and the experimenter explained what the boy and girl did to solve the example problem. In line with the adults studies of Torbeyns et al. (2009, 2011) the direct subtraction example applied the sequential direct subtraction strategy as taught in children's classrooms, characterized by sequentially taking away the hundreds, tens and units of the subtrahend from the minuend. Also in line with the adult studies of Torbeyns et al. (2009, 2011) the figure representing subtraction by addition used the complementary addition operation and first added from the subtrahend up to the next hundred, next up to the hundreds of the minuend and then up to the minuend, and in the last step all intermediate results were added to get the answer. Children were instructed to solve each problem with one of these two strategies.

In the *no-choice direct subtraction* condition, children had to solve all subtraction problems with the (sequential) direct subtraction strategy. We forced them to use this strategy via the experimental instruction. Likewise, in the *no-choice subtraction by addition* condition, all subtraction problems had to be answered via subtraction by addition. The mandatory use of subtraction by addition was again enhanced via the experimental instruction.

Procedure

All children were tested individually in a quiet room at their school. They all solved the three parallel series of problems in a within-subject design with different sessions on different days, and at least one day in between two sessions. All conditions were offered on a laptop computer. In each condition, children were told the strategy/strategies they were allowed to use by using the above-mentioned boy/girl figure(s). Afterwards, they were asked to solve one practice item for all allowed strategies. They were not allowed to use pen and paper. They had to verbally report their solution steps immediately after solving each subtraction problem to make sure that they understood the strategy/strategies. Next, a series of 13 trials (10 experimental items and 3 buffer items) was presented. Each trial started with an asterisk that appeared for 1000 ms in the centre of the screen, followed by the item presented in the same position. The characters were 2 × 2 cm large, black, separated by adjacent spaces and presented against a white background. The item remained on the screen until the child uttered the answer. Time started to run when the item appeared and ended when the experimenter pushed the spacebar of an external keyboard. Next, the experimenter entered the child's answer and the child was asked to verbally report his/her calculation steps. After this the next trial was initiated. No feedback was given. All conditions were audio-taped to allow a reliable analysis of the verbal strategy reports.

As recommended by Siegler and Lemaire (1997), all children started with the choice condition and continued with the no-choice conditions (within-subjects design). The order of the no-choice conditions was counterbalanced across children. Children were randomly assigned to one order of no-choice conditions. In each condition, children were instructed to solve all subtraction problems as accurately and as fast as possible and to verbally report the applied calculation steps immediately after solving each subtraction problem.

Analysis

Only the accuracy, speed and verbal report data on the 2040 SD and LD subtraction problems (10 items in 3 conditions for 68 participants) were included in the dataset. Before starting analyses we excluded 64 trials (3.14%) due to technical problems during data collection. Next, we coded all verbal strategy reports to check whether children only applied the strategy/strategies they were allowed to use. Children's verbal reports were coded as (a) direct subtraction, including all solutions in which the child sequentially took away the hundreds, tens and units of the subtrahend from the minuend; (b) subtraction by addition, including all reports in which the child used the complementary addition operation to solve the subtraction problem by first adding up from the subtrahend to the next hundred, then up to the hundreds of the minuend, then up to the minuend and finally computed the sum of all these intermediate results; or (c) other, consisting of all reports that could not be classified as direct subtraction or subtraction by addition as defined above and of all unclear strategy reports. Two researchers independently scored 10% of the strategy reports in the different conditions. They agreed on 85.38% of the trials and the interrater reliability was sufficient (Cohen's $\kappa = .75$). Trials with no agreement were reconsidered and in this second round consensus was found on all trials. No items had to be removed due to not following instructions (using direct subtraction in the no-choice subtraction by addition condition or vice versa). All 66 strategy reports classified as other strategies were removed from the dataset (3.30%). Finally, we controlled for whether participants (in all three conditions) had at least 2 out of 5 SD and LD subtraction problems remaining in the dataset. Five participants (two low achievers, two above-average achievers, one high achiever) for which this was not the case were removed from further analyses. The final dataset thus consisted of data of 63 participants (1829 items, 89.66% of the initial dataset).

The current dataset was analysed by using both frequentist and Bayesian statistics with JASP software (version 0.8.1.2), using the default JASP settings. We applied repeated measures ANOVA with Tukey-Kramer adjustments for post-hoc comparisons. In addition to these classic statistical analyses, we calculated Bayes Factors (BF), which quantified the evidence in the data for the alternative hypothesis (H_1) of a given effect compared to the evidence for the null hypothesis (H_0): BF_{10}. If BF_{10} is larger than 1, the data contains more evidence for H_1 than for H_0. If BF_{10} is between 1 and 0, the data contains more evidence for H_0. The size of the BF indicates the evidential strength for a given hypothesis, or stated differently, how many times more likely one hypothesis is than the other. We used Jeffrey's interpretation of BF, as clarified by Andraszewicz et al. (2015, Table 1). The BF can range from no evidence (close to 1) to decisive evidence (distant from 1, i.e., > 100 for H_1 and < 1/100 for H_0).

Results

Preliminary analyses involved the evaluation of order effects of the two no-choice conditions. Repeated measures ANOVAs revealed that there was no effect of order on the strategy distribution ($F(1,61) = 0.284$, $p = .596$), strategy efficiency (accuracy: $F(1,55) = 0.006$, $p = .937$; speed: $F(1,55) = 2.553$, $p = .116$) or strategy selection ($F(1,55) = 3.397$, $p = .071$). All BF_{10} values were below 1, indicating that the H_0 of no order effect was more likely (strategy distribution: $BF_{10} = 0.204$; strategy efficiency: accuracy: $BF_{10} = 0.273$, speed: $BF_{10} = 0.606$; strategy selection: $BF_{10} = 0.543$). This all indicated that the order of the no-choice conditions did not affect task performance. We grouped the data of both orders of no-choice conditions for further analyses. We present the results along the four parameters of the model of strategy change and investigate for each parameter how it was moderated by mathematics achievement level.

Strategy Repertoire and Distribution

In the choice condition, subtraction by addition was used at least once by most children. Only 9 children (14%) never used subtraction by addition; 17 children (27%) used it all the time; and 37 children (59%) used both direct subtraction and subtraction by addition. We observed no differences in strategy repertoire between the four achievement groups (Fisher's Exact $p = .43$), indicating that in all groups the vast majority of children reported the use of subtraction by addition at least once. Turning to strategy distribution, children more often applied subtraction by addition (M = 60%) than direct subtraction (M = 40%) to answer the subtraction problems in the choice condition, $t(62) = -2.08$, $p = .04$, partial $\eta^2 = .065$. The analysis of the BF, however, indicated that this should be interpreted with caution as the evidence for this difference was only anecdotal ($BF_{10} = 1.030$).

Strategy Efficiency

We analysed differences in the accuracy and speed (on both correctly and incorrectly answered items) of strategy execution between the no-choice conditions by means of repeated measurements ANOVA, with condition (no-choice direct subtraction vs. no-choice subtraction by addition) and problem type (SD vs. LD) as within-subject factors and mathematical achievement group (low, below-average, above-average, high) as a between subjects factor. Tables 2 and 3 present the accuracy and speed (on both correctly and incorrectly answered items) of strategy execution in the two no-choice conditions per subtraction type and achievement group.

Table 2

Accuracy (Percentage Correct) of Strategy Execution in the No-Choice Conditions per Subtraction Type and Achievement Group (Least Squares Means; SE)

	No-Choice Direct Subtraction				No-Choice Subtraction by Addition			
	SD		LD		SD		LD	
Achievement Group	***M***	***SE***	***M***	***SE***	***M***	***SD***	***M***	***SD***
Low	55.62	.10	75.83	.10	95.00	.06	71.87	.10
Below-average	82.33	.07	75.67	.07	96.00	.05	80.78	.07
Above-average	87.22	.07	70.83	.07	93.33	.04	73.33	.07
High	84.55	.06	71.59	.06	89.09	.04	75.45	.06
All	77.43	.04	73.48	.04	93.36	.02	75.36	.04

Focusing on *strategy accuracy*, a main effect of condition was found, $F(1, 59) = 9.31$, $p < .01$, partial $\eta^2 = .136$, $BF_{10} = 4.040$. Subtraction problems were more accurately solved when subtraction by addition had to be used (M = 84% correct) compared to direct subtraction (M = 75% correct). We also observed a very strong effect of problem type, $F(1, 59) = 15.91$, $p < .001$, partial $\eta^2 = .212$, $BF_{10} = 8414.022$, with SD subtraction problems (M = 85% correct) being more accurately answered compared to LD subtraction problems (M = 74% correct). There was a condition × problem type interaction, $F(1, 59) = 10.55$, $p < .01$, partial $\eta^2 = .152$, but the analysis of the BF indicated that this should be interpreted with caution as $BF_{10} = 0.273$.

There was no main effect of mathematical achievement group, $F(3, 59) = 0.37$, $p = .77$, partial $\eta^2 = .019$. The BF_{10} was equal to 0.110, which indicates substantial evidence for H_0 of no difference between the achievement groups. There was a condition × problem type × mathematical achievement group interaction, $F(3, 59) = 3.74$, $p = .02$, partial $\eta^2 = .160$, but the analysis of the BF indicated that this interaction should be interpreted with great caution as $BF_{10} = 0.067$.

Turning to *strategy speed*, we also found a main effect of condition, $F(1, 59) = 21.13$, $p < .01$, partial $\eta^2 = .264$, $BF_{10} = 365.314$. This indicates decisive evidence that subtraction problems were solved faster in the no-choice subtraction by addition condition (M = 14.15s) compared to the no-choice direct subtraction condition (M = 18.66s). There was an effect of problem type as well, $F(1, 59) = 119.74$, $p < .01$, partial $\eta^2 = .670$, $BF_{10} = 8.750 \times 10^{13}$. This provides decisive evidence that SD subtraction problems (M = 12.15s) were solved faster than LD subtraction problems (M = 20.66s). We also observed an effect of the condition × problem type interaction, $F(1, 59) = 60.54$, $p < .01$, partial $\eta^2 = .506$, $BF_{10} = 73239.045$. As shown in Table 3, SD subtraction problems were solved faster than LD subtraction problems in both the no-choice subtraction by addition condition ($p < .01$, partial $\eta^2 = .723$, $BF_{10} = 7996.836$) and the no-choice direct subtraction condition ($p < .01$, partial $\eta^2 = .213$, $BF_{10} = 168.493$). On the other hand, there was no difference in the LD subtraction problems in the two no-choice conditions ($p = .99$, partial $\eta^2 = .000$, $BF_{10} = 0.245$), whereas SD subtraction problems were solved faster in the no-choice subtraction by addition condition compared to the no-choice direct subtraction condition ($p < .01$, partial $\eta^2 = .487$, $BF_{10} = 2.091 \times 10^8$).

Table 3

Speed (Seconds) of Strategy Execution in the No-Choice Conditions per Subtraction Type and Achievement Group (Least Squares Means; SE)

	No-Choice Direct Subtraction				**No-Choice Subtraction by Addition**			
	SD		**LD**		**SD**		**LD**	
Achievement Group	***M***	***SD***	***M***	***SD***	***M***	***SD***	***M***	***SD***
Low	21.11	3.21	23.43	3.21	8.03	2.09	25.39	3.29
Below-average	17.12	2.35	24.41	2.35	7.58	1.53	21.13	2.40
Above-average	15.25	2.14	18.07	2.14	6.95	1.40	20.43	2.19
High	13.47	1.94	16.41	1.94	7.73	1.26	15.99	1.98
All	16.74	1.23	20.58	1.23	7.57	.80	20.73	1.26

We found no differences in speed in the no-choice conditions between the four mathematical achievement groups, $F(3, 59) = 2.11$, $p = .11$, partial $\eta^2 = .097$. An analysis of the BF indicated that this evidence was only anecdotal ($BF_{10} = 0.554$). There was an effect of the condition × problem type × mathematical achievement group interaction, $F(3, 59) = 3.07$, $p = .03$, partial $\eta^2 = .135$. The analysis of the BF, however, indicated that this finding should be interpreted with great caution ($BF_{10} = 0.221$).

Strategy Selection

In line with Torbeyns et al. (2011), we analysed children's strategy flexibility in two different ways. We first investigated whether children fitted their strategy choices to the numerical characteristics of the subtraction problems via a repeated measurements ANOVA with problem type (SD vs. LD) as within-subject factor, mathematical achievement group (low, below-average, above-average, high) as a between subjects factor and subtraction by addition frequency in the choice condition as the dependent variable.

There was an effect of problem type, $F(1, 59) = 41.34$, $p < .01$, partial $\eta^2 = .412$, $BF_{10} = 2.907 \times 10^7$. Children more frequently selected subtraction by addition on SD subtraction problems ($M = 3.73$) compared to LD subtraction problems ($M = 1.72$) and thus flexibly fitted their strategy choices to the numerical characteristics of the items. No main effect of achievement group was found, $F(3, 59) = 0.39$, $p = .76$, partial $\eta^2 = .019$, $BF_{10} = 0.104$ nor an interaction between achievement group and problem type, $F(3, 59) = 0.34$, $p = .80$, partial $\eta^2 = .017$, $BF_{10} = 0.066$. The BFs indicated that there was substantial evidence for the hypothesis of no difference between the achievement groups (H_0). This suggests that the achievement groups did not differ in their flexible application of subtraction by addition on the different problem types.

Second, we analysed whether children took into account their individual strategy efficiency competencies during the strategy selection process by correlating subjects' frequency of subtraction by addition in the choice condition with the accuracy and speed differences between the no-choice direct subtraction and no-choice subtraction by addition conditions. The correlation between children's subtraction by addition strategy frequency in the choice condition and the accuracy differences between the no-choice conditions was not significant, $r(63) = .00$, $p = .99$, $BF_{10} = 0.157$. These results indicate that children did not take into account their individual strategy accuracy characteristics during strategy selection. By contrast, they flexibly fitted their strategy choices to their individual strategy speed competencies, $r(63) = -.55$, $p < .01$, $BF_{10} = 6622.335$: The larger the speed difference between the no-choice subtraction by addition and the no-choice direct subtraction condition in favor of subtraction by addition, the more frequently children reported subtraction by addition in the choice condition. An ANCOVA on the accuracy and speed differences between the no-choice conditions, with the frequency of subtraction by addition in the choice condition and mathematical achievement group as predictor variables, indicated no differences between the four achievement groups. In line with the correlation analysis, we observed no effect of the frequency of subtraction by addition on the accuracy differences, $F(1, 55) = 0.00$, $p = .95$, $BF_{10} = 0.257$. We neither found an effect of mathematical achievement group, $F(3, 55) = 0.45$, $p = .72$, $BF_{10} = 0.210$. Although we observed an effect of the frequency of subtraction by addition on the speed differences, $F(1, 55) = 20.04$, $p < .01$, $BF_{10} = 4943.223$, there was no effect of mathematical achievement group, $F(3, 55) = 0.71$, $p = .55$, $BF_{10} = 0.227$. In all, both frequentist and Bayesian analyses indicated that there were no effects of mathematical achievement level on the children's strategy selection.

Conclusion and Discussion

Previous research with Lemaire and Siegler's (1995) model of strategy change and Siegler and Lemaire's (1997) choice/no-choice method revealed that adults frequently, efficiently and flexibly apply subtraction by addition on symbolically presented multi-digit subtraction problems (Torbeyns et al., 2009, 2011). However, in most countries current mathematics instruction strongly emphasizes direct subtraction (Haylock & Cockburn, 2003; Peltenburg et al., 2012; Selter et al., 2012; Torbeyns, De Smedt, et al., 2009a). The present study applied Siegler and Lemaire's theoretical and methodological tools to investigate the occurrence, frequency, efficiency and flexibility of using the subtraction by addition strategy in upper primary school children on subtraction problems in the number domain up to 1,000. To increase both our theoretical understanding of strategy acquisition in children of varying mathematical achievement levels and the potential educational implications of our study, we compared the various parameters of subtraction by addition strategy use among children along the broad continuum of mathematical achievement.

In contrast to previous studies on children's use of the subtraction by addition strategy, which followed a simple choice design (Blöte et al., 2000, 2001; De Smedt et al., 2010; Heinze et al., 2009; Selter, 2001; Torbeyns, De Smedt, et al., 2009a, 2009b), the present study not only revealed that the subtraction by addition strategy belongs to the strategy repertoire of primary school children, but also that they apply it frequently, efficiently and flexibly. Moreover, we observed hardly any differences among children of varying mathematical achievement levels, indicating that even below-average to low achievers, after a short explanation and demonstration of the subtraction by addition strategy, demonstrate mastery of the subtraction by addition strategy and are able to apply it frequently, efficiently and flexibly on multi-digit subtraction problems. These findings extend current insights into primary school children's strategies for doing symbolic multi-digit subtraction and challenge current primary mathematics instruction that strongly focuses on the routine mastery of direct subtraction.

Reporting Subtraction by Addition Strategy Use

In contrast with the strong instructional focus on direct subtraction and with the findings from previous studies with children (Blöte et al., 2000, 2001; De Smedt et al., 2010; Heinze et al., 2009; Selter, 2001; Torbeyns, De Smedt, et al., 2009a, 2009b), the vast majority of children in the present study reported the subtraction by addition strategy in the choice condition (= RQ1). A first possible explanation for this remarkable finding refers to the age of the children. Whereas previous studies with children reported on 2nd- to 4th-graders, we included 6th-graders. They all had practiced multi-digit subtraction problem solving for more than three years, as evidenced by our textbook analyses and teacher interviews, and (thus) presumably mastered the explicitly taught direct subtraction strategy sufficiently well to be able to invent other strategies, including the subtraction by addition strategy (Shrager & Siegler, 1998). So, the availability of the subtraction by addition strategy in our sample's strategy repertoire might be due to children's greater experience with multi-digit subtraction problems and consequently their greater capacities to invent alternative strategies. Future studies - conducted along the same theoretical and methodological framework as the present one - in younger children at the start of multi-digit subtraction instruction are needed to evaluate this hypothesis. A second explanation for the availability of subtraction by addition in most children's strategy repertoire deals with the experimental design. Contrasting previous studies with children, we explicitly mentioned the subtraction by addition strategy as a possible strategy to solve multi-digit subtraction problems at the onset of the interview. As children might have serious difficulties in verbally reporting an untaught arithmetic strategy (e.g., Peters, De Smedt, Torbeyns, Ghesquière, & Verschaffel, 2012, 2013), the demonstration and articulation of this strategy by the experimenter at the beginning of the choice session might have helped children to (adequately) verbally report their subtraction by addition strategy use. Moreover, the explicit communication of subtraction by addition as a possible alternative strategy might have established a socio-mathematical norm that stimulated children to report an available but not explicitly taught strategy (Yackel & Cobb, 1996).

Frequency, Efficiency and Flexibility of Subtraction by Addition

Our study demonstrated not only that the subtraction by addition strategy was part of children's strategy repertoire, but also that they applied the untaught subtraction by addition strategy frequently and efficiently (= RQ2 and RQ3). Children solved multi-digit subtraction problems surprisingly efficiently via subtraction by addition. Despite the highly frequent practice of direct subtraction in children's classrooms, they were slower in executing this strategy compared to the untaught subtraction by addition strategy in the no-choice conditions. Moreover, they did not only answer small-difference subtraction problems but also large-difference subtraction problems faster via subtraction by addition than via direct subtraction. This finding challenges current mathematics instruction, which capitalizes on the routine mastery of direct subtraction and pleas for more instructional attention to subtraction by addition as alternative - and maybe even more obvious - strategy in this domain (cf. Peltenburg et al., 2012; Selter et al., 2012). As this study is the first to analyse children's accuracy and speed of direct subtraction versus subtraction by addition execution on multi-digit subtraction problems following a choice/no-choice method, future studies are needed to replicate and refine our findings also in younger age groups, before such drastic changes in current mathematics instruction can be advised. Moreover, referring to the two-dimensional scheme for classifying multi-digit subtraction strategies introduced at the beginning of this article (see Peltenburg et al., 2012; Torbeyns, Ghesquière, et al., 2009), our study involved only one type of direct subtraction and subtraction by addition strategy, namely the sequential type and more specifically the specific version of the sequential direct subtraction strategy that is currently taught as the default strategy for doing multi-digit subtraction in Flemish classrooms and its most appropriate counterpart for doing sequential subtraction by addition. Future studies are required to explore the superior efficiency of subtraction by addition over direct subtraction for other general and specific types of direct subtraction and subtraction by addition too (Peltenburg et al., 2012).

In addition to these further descriptive studies on the superior efficiency of subtraction by addition compared to direct subtraction, it is also of utmost interest to systematically investigate the reasons for these observed differences in efficiency. As a first explanation, we refer to the intrinsic difficulty of the underlying addition and subtraction processes. It can be argued that the process of adding a number to a given number is inherently easier than the process of taking away a similar number from a given number. Although it is difficult to provide direct evidence for this claim, there are several elements from various sources in support of it (see also Torbeyns et al., 2011). First, from a mathematical perspective, the additive operation is fundamental for the development of the natural numbers, the most basic number system (Kilpatrick et al., 2001). Natural numbers are constructed on the basis of the principle of recurrence, which means that every natural number is constructed on the basis of the previous number as the successor of *n* (Burrill, 1967; Dedekind, 1963; Ifrah, 1985; Mainzer, 1995). In other words, the system of natural numbers is created and defined in an additive order, almost automatically leading to the priority of addition to subtraction. Second, there is ample evidence in the psychological literature on early mathematics for young children's earlier mastery of and better performance on counting up compared to counting down, which can be considered as important precursors of the processes of adding and subtracting, respectively (Fuson, 1986, 1992; Verschaffel et al., 2007). Third, many psychological studies have shown that addition facts are acquired quicker and solved more efficiently than the corresponding subtraction facts (Campbell & Xue, 2001) and that many people solve single-digit subtraction facts by relying on the complementary addition fact (Campbell, 2008). Fourth, the priority of addition to subtraction is also reflected in children's curricula, textbooks and teacher practices for elementary arithmetic. Worldwide, mathematics teachers first teach their pupils how to add numbers and only afterwards - after sufficient practice and mastery of the addition operation - teach how to subtract (Kilpatrick et al., 2001). The earlier and more intensive practice of the addition operation in the classroom may further contribute to addition being more familiar and easier than subtraction.

A second explanation for the efficiency of subtraction by addition that requires further research attention relates to the number of carry and borrow operations that had to be performed when using subtraction by addition or direct subtraction in the present study. Cognitive psychological studies revealed that the number of trades influences the accuracy and speed of responding in complex addition and subtraction (Ashcraft & Kirk, 2001; Geary, Frensch, & Wiley, 1993; Imbo, Vandierendonck, & De Rammelaere, 2007; Imbo, Vandierendonck, & Vergauwe, 2007): Performance becomes slower and less accurate as more carry or borrow operations need to be performed, which might be explained in terms of the demands these operations put on people's working memory. In the present study, all subtraction problems required two crossing-overs. So, when using the direct subtraction strategy, children inevitably had to execute two borrow operations: They had to borrow both from the hundreds and the tens. By contrast, with the variant of subtraction by addition used in the present study, no carry operations were needed. The inclusion of different types of subtraction problems, including subtraction problems without crossing-overs such as 628 – 313 = __ can help to address this second explanation in future studies on the topic.

All children also flexibly fitted their strategy choices to both numerical characteristics of the subtraction problems and to their individual strategy speed competencies (= RQ4). This suggests that they were also able to demonstrate some flexibility in their use of the subtraction by addition strategy, despite the absence of any instructional attention to it. This finding complements previous results on strategy flexibility in single-digit addition and subtraction, evidencing more flexible strategy choices with increasing experience in the domain (Siegler, 1996, 2000). Moreover, it significantly adds to our understanding of the structures and mechanisms underlying children's strategy choices on multi-digit subtraction problems using an extended definition of strategy flexibility as fitting strategies not only to the numerical characteristics of items (Blöte et al., 2000, 2001; De Smedt et al., 2010; Heinze et al., 2009; Peters et al., 2012, 2013; Selter, 2001; Torbeyns, De Smedt, et al., 2009a, 2009b), but also to individual strategy efficiency competencies (Verschaffel, Luwel, Torbeyns, & Van Dooren, 2009).

Comparing Children of Varying Mathematical Achievement Level

The present study revealed that both mathematically higher achieving children and their lower achieving peers frequently, efficiently and flexibly applied the untaught subtraction by addition strategy to solve multi-digit subtraction problems (RQ5). Just like their above-average and high achieving peers, low and below-average achievers solved multi-digit subtraction problems frequently and efficiently via subtraction by addition. Moreover, even the lower achieving children flexibly fitted their strategy choices to both numerical characteristics and individual strategy competencies. This is in sharp contrast with the assumption that the acquisition of strategy flexibility is most difficult for the lower achieving children (Geary, 2003; National Mathematics Advisory Panel, 2008; Verschaffel et al., in press).

The results of the present study are in line with previous findings on low achievers' subtraction by addition strategy use (Peltenburg et al., 2012; Peters, De Smedt, Torbeyns, Ghesquière, & Verschaffel, 2014). They also complement and extend these findings by using a choice/no-choice method that systematically addresses this strategy's occurrence, frequency and efficiency and by applying a more complex definition of strategy flexibility as fitting strategy choices to both item and strategy performance characteristics. Together with these earlier findings, our results indicate that with sufficient practice in and mastery of multi-digit subtraction problem solving, even low achieving children seem able to efficiently and flexibly use alternative strategies. These findings support the feasibility and value of teaching for strategy variety and flexibility in children of all achievement levels (Baroody, 2003; Kilpatrick et al., 2001; Verschaffel et al., 2007; Verschaffel et al., in press). However, as none of the children participating in the present study were known in the school as being classified as having a mathematics learning disability, it is a challenge for future research to investigate the occurrence, frequency, efficiency and flexibility of subtraction by addition versus direct subtraction using a choice/no-choice design in this group as well: children with mathematical disabilities. As the present study only included children with ample experience in the domain of multi-digit subtraction (see also above), it is important that future studies also address the strategy competencies of children of varying mathematical achievement levels at the start of instruction. The comparison of the occurrence, frequency, efficiency and flexibility of subtraction by addition versus direct subtraction across achievement groups at both the start and the end of multi-digit subtraction instruction will help to understand the feasibility of strategy variety and flexibility in children of all achievement levels along the whole instructional cycle.

Educational Implications

Combining theoretical and methodological tools from cognitive psychology and insights from mathematics education research, the findings of the present study raise questions about the adequacy of current mental multi-digit subtraction instruction with its strong focus on direct subtraction. Despite the strong instructional effort for mastering direct subtraction and the intensive practice of this strategy in classrooms, children answered multi-digit subtraction problems more efficiently with the untaught subtraction by addition strategy than with the explicitly taught and intensively practiced direct subtraction strategy. Moreover, and in line with the adult studies of Torbeyns et al. (2009, 2011), subtraction by addition proved to be highly efficient not only on small-difference but also on large-difference subtraction problems. As outlined above, future studies are needed to further analyse the efficiency of subtraction by addition in younger age groups and children with mathematical difficulties and on all types of multi-digit subtraction problems.

In addition to ascertaining studies, we need intervention studies wherein new instructional approaches to the introduction and development of the subtraction by addition strategy are designed, implemented and tested. In this respect, one should be aware that the categorisation of multi-digit subtraction strategies on the basis of their underlying operation (differentiating between direct subtraction and subtraction by addition) might be less intuitive for children - especially for the younger and lower achieving ones - than their categorisation in terms of the concrete solution steps involved in the solution process (differentiating between decomposition, sequential and varying strategies). However, the mathematics education literature already contains several appealing proposals for introducing the former classification to children, such as the inclusion of clear external models to represent the different strategies including the empty number line. These proposals and models might serve as promising didactic tools to foster children's understanding of the differences between and value of the different strategies (e.g., Beishuizen, 1997; Müller, Selter, & Wittmann, 2012) and, consequently, stimulate the acquisition of strategy variety and flexibility as major aims of primary mathematics education.

Notes

i) It should be noted here that the operation perspective distinguishes also a third type of strategy, namely indirect subtraction. In case of indirect subtraction, one finds the solution by determining how much has to be decreased or subtracted from the minuend to get the subtrahend (e.g., solving 628 – 313 = __ via 628 - ? = 313) (De Corte & Verschaffel, 1987).

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Competing Interests

The authors have declared that no competing interests exist.

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