Fraction Errors in a Digital Mathematics Environment: Latent Class and Transition Analysis

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Abstract

Student struggles with fractions are well documented, and due to fractions' importance to later mathematics achievement, identification of the errors students make when solving fraction problems is an area of interest for both researchers and teachers. Within this study, we examine data on student fraction problem errors in pre- and post-quizzes in a digital mathematics environment. Students (N=1,431) were grouped by prevalence of error types using latent class analysis. Three different classes of error profiles were identified in the pre-quiz data. A latent transition analysis was then used to determine if class membership and class structure changed from pre- to post-quiz. In both pre- and post-quiz, there was a class of students who appeared to be guessing and a class of students who performed well. One class structure was consistent with the idea that early fraction learners rely heavily on whole number principles. Identification of co-occurrence of and changes to fraction errors has implications for curricular design and pedagogical decisions, especially in light of movements toward personalized learning systems.

Keywords: fraction errors, elementary mathematics, latent class analysis, latent transition analysis

1 Introduction

Educators have long been concerned with students' struggles with fractions (Brown & Quinn, 2012). An analysis of the 1990 National Assessment of Educational Progress revealed that less than half of 12th graders were successful with fractions, percentages, decimals, and simple algebra (Mullis, Dossey, Owen, & Phillips, 1991). These rates have not improved in more recent years across a number of U.S. states (e.g., California Department of Education, 2008; Higgins, 2008; Kim, Schneider, Engec, & Siskind, 2006; Pennsylvania Department of Education, 2011). Struggle with fractions is especially concerning, because fraction knowledge is important for later success in mathematics (NMAP, 2008). In a study by Siegler and colleagues (2012), fraction and whole-number division knowledge were the strongest predictors of both algebra achievement and overall mathematics achievement five to six years later, even after controlling for a host of demographic and prior achievement factors. Similarly, Bailey, Hoard, Nugent, and Geary (2012) found that fraction achievement in sixth grade predicted general mathematics achievement in seventh grade, also after controlling for demographic and prior achievement. Additionally, fraction knowledge has been shown to be a stronger predictor of future mathematics achievement than whole number knowledge (e.g., Bailey et al., 2012; Booth & Newton, 2012; Siegler et al, 2012). This supports the hypothesis that fraction skills are among the early math skills more predictive of later mathematics achievement (e.g., Bailey et al., 2012).

With fractions critical to later mathematics achievement, it is important to understand some of the reasons students struggle with fractions. One way to do this is through understanding the common errors students make when solving fraction and

other mathematics problems (Bottge, Ma, Gassaway, Butler, & Toland, 2014; Voza,

2011). These error analyses can help researchers and educators address fraction

misunderstandings by providing a target for mathematics interventions. With this paper,

we extend the analysis of fraction errors to a digital mathematics environment—a type

of instructional tool that is gaining increasing prevalence in the modern classroom

(Buckingham, 2013). We identify which types of errors students make within the Spatial

Temporal (ST) Math elementary mathematics learning software and whether students

can be grouped according to these errors and their co-occurrence, and examine how

these groupings change after students have been exposed to fraction instruction within

the platform.

1.1. Why Do Students Struggle with Fractions?

Prior research has offered a number of reasons why students struggle with

fractions: fractions are multifaceted (e.g., Pantziara & Philippou, 2012), fraction

algorithms are far more complex than natural numbers (e.g., Hiebert, 1992; Stafyliduo &

Vosnidou, 2004), and students often experience a gap between conceptual and

procedural knowledge (e.g., Bailey, Siegler, & Geary, 2014; Jordan, Hansen, Fuchs,

Siegler, Gersten, & Micklos, 2013; Jordan, Resnick, Rodrigues, Hansen, & Dyson,

2016). The following sections explain how these factors contribute to student difficulty

with fraction understanding.

1.1.1 Fractions are multifaceted. Fractions are thought to have five

subconstructs: part-whole, ratio, quotient, operator, and measure—also known as

magnitude (Charalambous & Pitta-Pantazi, 2007; Pantziara & Philippou, 2012). Each of

these subconstructs represents a different aspect or way to use fractions. See Table 1

for descriptions of each construct.

Table 1
Sub-Constructs of Fractions

Sub-construct	Definition (Charalambous & Pitta-Pantazi, 2007)	Example with 2/3
Part-whole	an object is partitioned and the fraction compares the number of parts to the total number of partitions	two out of three slices of pizza
Ratio	comparison of two different quantities	two slices of cheese pizza and three slices of pepperoni
Operator	function applied to number, object, or set	two-thirds of a pizza, regardless of the number slices
Quotient	division	splitting two pizzas between three people
Measure/ magnitude	does not fit easily into the pizza example because it represents where a fraction is placed on a number line (or how big the fraction is)	

Although many children may understand fractions as a part-whole, this concept

alone is not enough for complete understanding of fractions (Lamon, 2012; Pantziara &

Philippou, 2012). Charalambous and Pitta-Pantazi (2007) included knowledge of a

fraction's measure as necessary for developing proficiency in fractions and noted that

performance on fraction measure tasks is poor compared to performance on tasks that

represent the other constructs. One reason may be because having a two-dimensional

number makes it harder for students to grasp the one-dimensional concept of measure

(sometimes referred to as magnitude, e.g., Booth, Newton, & Twiss-Garrity, 2013).

Needing to understand how two numbers (the numerator and denominator) can be

condensed into a single number or one-dimensional point on the number line makes it

harder for students to perform simple algorithms, such as ordering and comparing

fractions (see the integrated theory of numerical development, Rinne, Ye, & Jordan,

2017; Siegler, Thompson, & Schneider, 2011). As in Table 1, consider 2/3 and compare

it to 5/6. It is difficult to immediately know which fraction is larger. Seeing that both five

and six are larger than two and three, students may believe that 5/6 is larger. However,

students are taught that the bigger the denominator, the smaller the part. This may lead

to students saying 5/6 is the smaller number because it has the bigger denominator. It is

not until we look at the relationship between two and three (for 2/3) and five and six (for

5/6), that we can be sure that 5/6 is larger.

Table 2

Comparison of Fractions and Natural Numbers

	Natural Numbers	*Fractions*	*Fraction Arithmetic*
Form	Takes the form ab	Takes the form a/b	
Ordering	Ordering number depends on comparing similar place values	Ordering depends on the relationship between numerator and denominator	
Quantities	Discrete	Discrete and continuous	
Addition/ Subtraction	Combining digits of same place value	Use equivalent fractions with common denominator then combine numerators	$a/b + c/d = da/db + bc/db = (da+bc)/db$
Multiplication	Multiplication makes number bigger	Multiplication/division makes number bigger or	$a/b \times c/d = ac/bd$
Division	Division makes number smaller	smaller depending on the fractions	$a/b \div c/d = a/b \times d/c = ad/bc$

Note. Differences as stated in prior research.

    1.1.2. Fraction arithmetic is more complex than natural numbers. The

algorithms used for addition, subtraction, multiplication, and division are also more

complex for fractions than for natural numbers. When adding or subtracting natural

numbers, one can simply combine the digits of the same place value and regroup if

necessary. However, when adding or subtracting fractions, one must first find the least

common denominator, transform the fractions so they will have the same common

denominator, then add/subtract the numerators. Table 2 further describes how

arithmetic (and their algorithmic symbols) differ for fractions.

The typical order of mathematics instruction may compound student difficulties in understanding fraction complexity. In most curricula, students learn natural numbers and then fractions (e.g., Common Core State Standards Initiative, 2016). Although learning fractions after learning natural numbers is a logical sequence, students often have a hard time reconciling the schema of natural numbers with that of fractions (Hiebert, 1992; Mack, 1995; Stafyliduo & Vosnidou, 2004). This may be especially problematic because algorithmic symbols for fractions do not have the same meaning as the algorithmic symbols for whole numbers (see Table 2). The change in the procedures associated with these symbols can be confusing for children who need to modify their schema for the symbols (e.g., the procedure for adding natural numbers compared to the procedure for adding fractions; Hiebert, 1992, p. 294). Thus, it is not just the concepts of fractions that are complex, their procedures are complex as well and often do not correspond to the procedures used with whole numbers.

1.1.3. Conceptual versus procedural knowledge. Each of the aforementioned challenges to understanding fractions highlight the difficulties students may have in gaining conceptual knowledge of fractions. Conceptual and procedural knowledge have a bidirectional relationship, but are separate entities that contribute to fraction success (Baroody & Ginsburg, 1986; Rittle-Johnson & Alibali, 1999; Siegler & Stern, 1998). Thus, researchers sometimes split fraction knowledge into concepts and procedures (e.g., Bailey, Siegler, & Geary, 2014; Jordan et al., 2013; Jordan et al., 2016). Conceptual knowledge is the "explicit or implicit understanding of the principles that govern a domain and of the interrelations between pieces of knowledge in a domain" (Rittle-Johnson & Alibali, 1999, p. 175). By comparison, procedural knowledge is "the

knowledge of how to perform mathematical tasks…meant to generate the right answer to a given type of problem" (Hallet et al., 2012, p. 470). Although both conceptual and procedural knowledge are important to fraction achievement, reliance on procedural knowledge alone is unlikely to result in true understanding and advanced achievement—students who rely primarily on procedural knowledge may apply it incorrectly because they lack the conceptual knowledge that tells them when and why to use specific procedures (Hallet, 2008).

1.2. What Types of Errors Do Students Make?

It is important to understand both *why* students struggle with fractions and *how* they struggle with fractions. To understand how students struggle with fractions, researchers have looked at the errors students make when solving fraction problems (e.g., Ashlock, 2001; Bottge et al., 2014; Brown & Quinn, 2006; Malone & Fuchs, 2016). To do this, researchers give participants a test/activity and observe errors made. Most tests given in this manner have been free response (e.g., 1/2 + 3/4 = _____), although some have included multiple choice (e.g., Brown & Quinn, 2006). Although this procedure is used among most fraction error researchers, there are differences in methods in terms of what skills are tested and how errors are addressed. For example, some researchers have focused on specific fraction knowledge, like ordering or adding/subtracting fractions (Bottge et al., 2014; Malone & Fuchs, 2016), whereas others have focused on a broad range of fraction knowledge (Ashlock, 2001; Brown & Quinn, 2006). Additionally, some researchers define categories of errors a priori and then examine tests to see how often students make these errors (Bottge et al., 2014; Malone & Fuchs, 2016), whereas other researchers first give tests and then code and

define errors from student answers (Ashlock, 2001; Brown & Quinn, 2006). Regardless

of the method used, two broad categories of fraction errors have emerged, general

misunderstanding of the concepts of fractions and improper application of whole

number rules to fraction problems (e.g., Ashlock, 2001; Brown & Quinn, 2006; Bottge et

al., 2014; Malone & Fuchs, 2016). These a priori errors and their origins will be further

discussed in the context of the results of this study (section 3.1. Fraction Errors).

1.3. Motivation for Current Study

There is an extant body of work identifying student errors in fraction problems

and examining why students might make these errors (e.g., Ashlock, 2001; Brown &

Quinn, 2006). Although these studies have contributed to our understanding of the

structure of student struggles with fractions, they have largely viewed student errors in

isolation, without consideration of the instructional context. Work examining student

fraction errors using assessments embedded in authentic educational contexts might

provide information more immediately applicable to the classroom. Additionally,

previous research has used student errors to enhance instruction (e.g., Ashlock, 2001),

but often focuses on individual errors and what they mean. However, it is likely that

students who make one type of error also make another type of error. By classifying

error profiles, teachers may be able to more efficiently address a combination of errors.

Finally, examining profiles of errors both before and after an instructional event may

reveal both typical patterns of change and those errors and patterns that may be

resistant to instruction. This can lead to refinement in instruction aimed at common

profile groups and the identification of smaller groups of students for more targeted

intervention. Similar research has been conducted to see what algebraic errors are

persistent throughout instruction, but this has not yet been extended to fractions (Booth, Barbieri, Eyer, & Paré-Blagoev, 2014).

1.4. Current Study

Using data from within the digital mathematics software ST Math, we identify common fraction errors, examine co-occurrence of errors to define student error profiles, and investigate the changes in profiles from before to after instruction. Specifically, we ask:

1. What errors are made when students solve fraction problems within ST Math?

2. How can students be grouped into classes by the errors they tend to make together? How do student characteristics differ between these classes?

3. How do these classes change from pre-quiz to post-quiz?

We expect that certain errors will be more prevalent in the pre-quiz due to a naïve understanding of fractions, such as the ratio error or whole number ordering (Ashlock, 2001; Malone & Fuchs, 2016). We also expect errors related to fraction magnitude to be persistent into the post-quiz due to the construct's complexity (Booth et al., 2013; Rinne et al., 2017; Siegler, et al., 2011).

2 Method

2.1. Context: ST Math

ST Math, created by MIND Research Institute (MIND), is an interactive instructional software for computers and tablets that is based on theory that suggests that the ability to visualize mathematics concepts leads to better conceptual knowledge and performance (see Geary, 1995; National Research Council, 2005; Shaw & Peterson, 2000). ST Math has previously been shown to result in small improvements in

math achievement, especially on topics involving number sense (Rutherford et al.,

2014; Schenke, Rutherford, & Farkas, 2014). ST Math is currently used in 45 states with

over one million students and is designed to align with both Common Core and relevant

state standards, including those in Florida, the site of this study.

  Within ST Math, students progress through a number of objectives focused on

specific math concepts. The ST Math content follows a hierarchical pattern: objective,

sub-objective, game, level, puzzle. Within the objective/sub-objective, there are a

variety of games that use the same imagery and design throughout their levels. Each of

these games contains between one and 10 levels that increase in difficulty. Within each

level, students complete interactive puzzles, which are the delivery method for the

mathematics content. For each level, the student has between one and three lives—if

they answer more puzzles incorrectly than they have lives, they are removed from the

level and can chose to replay it or replay a previously-passed level. Before students

begin the objective, they must complete a five-question multiple choice pre-quiz on that

objective's content. After demonstrating mastery of the content within an objective by

successfully completing all levels within, the student then completes a five-question

multiple choice post-quiz that mirrors the pre-quiz, question-for-question in topic, but

uses different specific examples and numbers. The pre- and post-quizzes have either

three or four answer choices for each question (see Table 5 on page 15 and Figure 1 on

page 25 for examples).

  The third-grade curriculum for the 2015-2016 Florida version of ST Math had 23

required objectives and eight optional objectives. These objectives covered content

such as place values, addition and subtraction, multiplication and division, and fractions.

The three objectives covering fraction content were part of the required curriculum and

began approximately 65% of the way through the curriculum.

Table 3
Demographics

Variable	Percent of Sample (N=1,431)	Percent of Total (N=4,290)
Male	54%	52%
Student with Disability	10%	17%*
Free/Reduced Lunch	46%	58%*
English Language Learner	11%	13%*
Gifted	19%	13%*
Race		
Asian	6%	4%*
Black	13%	21%*
Hispanic	16%	18%*
White	60%	53%*
Other	5%	4%

Note. Differences between total and analysis samples were determined using chi-squared tests.
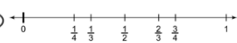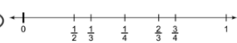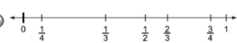There were 75 schools in the sample and total.
*Statistically significantly different at $p < .05$.

2.2. Participants

Participants were third graders from a school district in central Florida

participating in an NSF-funded project relating gameplay within ST Math to student

achievement and motivation. Students within the district played ST Math as part of their

normal instruction during the 2015-2016 school year. The sample (N = 4,290) was

limited to third graders with pre- and post-quiz data from the three fraction objectives (N

= 2,187), and who also had district-provided demographic and achievement data (N =

1,431). The 1,431 students attended 75 different schools, 54% were male, 60% were

white, 16% were Hispanic, 13% were black, 5% were Asian, and 46% qualified for free

or reduced lunch (see Table 3). The analysis sample differed at statistically significant

levels (*ps* < .04) on most demographics from the larger sample of third graders. There

were fewer students with disabilities, who qualified for free/reduced lunch, were English

Language Learners, and who were black. Conversely, there were more gifted students,

Asian students, and white students. Because the sample was limited to students that

had completed all three fraction objectives, the reduced sample may have a lower

percentage of students using ST Math for remedial reasons.

Table 4
ST Math Fraction Objective for Third Graders

Quiz	Question Types	Example Question
Fraction Concepts (64% completion rate)	1. match a fraction figure to a written fraction 2. match whole numbers to their equivalent fractions (e.g., 3 = 3/1)	Which fraction describes the shaded region? (A) $\frac{2}{5}$ (B) $\frac{2}{3}$ (C) $\frac{4}{3}$ (D) $\frac{5}{2}$ First type
Fractions on the Number Line (60% completion rate)	1. match a point on a number line to a fraction 2. match three fractions to points on the number line 3. match a fraction to a point on the number line (possible points labeled with letters)	Which number line shows the correct information for $\frac{1}{2}, \frac{1}{4}$ and $\frac{1}{3}$? Second type

Comparing Fractions (54% completion rate)	1. match a fraction figure to a written fraction
	2. match three fractions to points on the number line
	3. identify the correct statement of magnitude comparisons
	  a. identify the correct fraction sentence
	  b. select the appropriate fraction to complete the number sentence
		$\frac{1}{2} >$ ___
		(A) $\frac{1}{1}$ (B) $\frac{3}{4}$ (C) $\frac{2}{1}$ (D) $\frac{1}{8}$
		Third (b) type

2.3. Measures of Fraction Errors

As implemented in Florida during the 2015-2016 school year, the ST Math

curriculum aligned with the Florida Standards, which introduced fractions in the third

grade (Mathematics Florida Standards, 2014; MIND Research, 2016). The three fraction

objectives within the ST Math third grade curriculum during this year were Fraction

Concepts, Fractions on the Number Line, and Comparing Fractions. The pre- and post-quizzes for these objectives tested fraction concepts and algorithms. Table 4 describes the types of questions within each objective's quizzes.

2.4. Procedure

2.4.1. Error coding. Data were the answer choices on the pre- and post-quizzes on fraction objectives. Only incorrect answer choices were included in the analysis of errors. Overall, there were 86 possible incorrect answer choices for the six quizzes (pre- and post-quizzes for three objectives). Error coding was completed in three key steps. First, answer choices were qualitatively coded using a priori and a posteriori codes. The a priori codes were taken from previous research on fraction errors (e.g., Ashlock, 2001; Bottge et al., 2014; Brown & Quinn, 2006; Malone & Fuchs, 2016). When errors did not align with a priori codes, they were given an a posteriori code based on the presumed logic used to select that answer. After the first round of coding was finished, the authors collaborated with personnel at MIND to confirm or refine the researchers' a posteriori codes. In designing the quiz questions, MIND often crafted incorrect answer options to represent common errors they had observed students make in prior versions of ST Math; thus, these common errors from MIND's observations were reflected in the final codes. Lastly, error code names were refined to follow consistent naming conventions. Even after the three rounds of coding, there were still some answer choices that did not fit with a known or postulated error pattern. These answer choices were deemed filler (or random) choices. For example, in a question asking what fraction represented a circle with five parts that has one part shaded (e.g., 1/5) the answer option 2/5 did not follow any identifiable error logic. Thus, this answer choice was coded as "filler." From

the 86 incorrect answer choices, there emerged eight different error types and the filler

choice, for a total of nine error categories.

2.4.2. Data cleaning. To make the error codes usable data, several steps were

followed in the statistics analysis software *Stata* (version 14; StataCorp, 2015).

Variables were created that represented the proportion of times the student made that

specific error to the total errors the student made. In this manner, if a student made the

complement error twice and the filler error three times, they would have a total of five

errors. The proportion for the complement error would be 0.4 (2/5) and the proportion

for the filler error would be 0.6 (3/5). All other errors would have proportions of zero

because they did not make those errors. The last step was creating an ordinal scale that

would be compatible with latent class and transition analyses (Goodman, 2002):

0. the student did not make the error;

1. the student made the error between 0% and 25% of the time ($0 < x \leq 0.25$);

2. the student made the error between 25% and 50% of the time ($0.25 < x \leq 0.50$);

3. the student made the error between 50% and 75% of the time ($0.50 < x \leq 0.75$);

4. the student made the error between 75% and 100% of the time ($0.75 < x \leq 1$).

Error proportions were used in analyses instead of number of errors made to control for

the number of errors students made overall. Using number of errors would bias results

by heavily weighting on student performance and we would lose the relative frequency

of each error.

2.4.3. Latent class analysis. Latent class analysis (LCA) is a mixture modeling

method used with categorical variables. It attributes the relationship between variables

to an unobserved, latent variable (Collins & Lanza, 2010; Goodman, 2002; Nylund,

Asparouhov, & Muthen, 2007). The goal of LCA is to group people into classes based on their observed variables. These classes represent the underlying categorical latent variable. This procedure is similar to factor analysis; however, it provides a person-centered approach instead of a variable-centered approach. Thus, the classes are created to group similar participants together, rather than similar variables (Collins & Lanza, 2010; Goodman, 2002). The person-centered approach allows for generalizations of the patterns of behavior. In this study, LCA was used to determine fraction error patterns. Models with *k* versus *k + 1* classes were tested iteratively to determine best fit. To determine best fit, sample size adjusted Bayesian Information Criterion (adjusted BIC), Bayes Factor, Entropy, and the bootstrap likelihood ratio test (BLRT) were used (Nylund et al., 2007). After conducting the LCA, class membership was regressed on demographics and game-play variables to further understand the composition of each class and external variables that predicted class membership. For this step, logistic regressions were used and were clustered at the teacher level to account for nesting within classrooms.

      2.4.4. Latent transition analysis. To determine if class membership changed between time points, such as between a pre- and post-quiz, latent transition analysis (LTA) was used (Rindskopf, 2010). Rindskopf (2010) defines LTA

      as a statistical model in which (i) latent categorical constructs are defined at two

      or more time points, (ii) parameters are included that assess initial status and

      transition probabilities from time *i* to *i* + 1..., and (iii) observed variables are

      imperfect indicators of the hypothesized latent variables. (p. 199)

LTA regresses the class variable at the second time point onto the first time point to

determine the likelihood of the classes remaining the same. LTA was used to determine

if classes remained the same between pre- and post-quiz. The *Mplus* software (version

7; Múthen & Múthen, 2016) was used to run both the LCA and LTA.

Logistic regressions were used to determine how students differed between the

classes, as was done with pre-quiz classes. Additionally, logistic regressions were used

to predict the movement from pre- to post-quiz class. All logistic regressions were

clustered at the teacher level to account for classroom similarities.

3 Results

3.1. Fraction Errors

Three of the 10 error codes were developed a priori—illogical sizing/spacing

(Ashlock, 2001), ratio (Ashlock, 2001), and whole number ordering (Malone & Fuchs,

2016)—and the other six were a posteriori—complement, filler, incomplete information,

reciprocal, reducing fractions to whole numbers, and same numerator/denominator

ordering. Table 5 summarizes the types of questions and the errors possible within each

question type. Overall, students made an average of seven errors on pre-quizzes and

four errors on post-quizzes (out of 15 questions each for both pre- and post-quizzes,

see Table 6). Table 7 details the proportions of times each error was made and how

many students made it. See Figure 1 for example of errors.

3.1.1 Complement. The complement error occurred when the student chose the

complementary fraction to the correct answer choice. For example, if the question

asked what fraction represents the shaded part of a shape and the answer was 2/3, the

complement would be 1/3. On average, students made this error 11% of the time in pre-

quizzes and 9% of the time in post-quizzes. About 29% of students made the complement error in pre-quizzes and 23% in the post-quizzes.

3.1.2. Filler. The filler error encompassed answer choices that did not appear to follow a known error logic. Consider the example in Figure 1. The correct answer was 0/1 and 1/1 was coded as representative of reducing fractions to whole number error. However, 1-0 and 0-1 were coded as filler because they did not fit with any other error and they addressed whole number subtraction more so than fractions. For the pre-quiz, there were six questions that contained at least one filler error, with eight answer choices coded as filler. For the post-quiz, there were four questions that contained at least one filler error, with six answer choices coded as filler. The filler error was made about 11% of the time on pre-quizzes and 7% of the time on post-quizzes. Approximately 49% of students made this error at least once on pre-quizzes and 55% on post-quizzes.

3.1.3. Illogical sizing/spacing. This error represented misunderstanding that fraction parts must have equal sizes and consistent spacing (Ashlock, 2001). There were two ways this error was reflected. The first was when students were presented with figures divided into unequal parts and were asked to match this figure to a fraction. Some students' answers included the number of (unequal) parts in the denominator, indicating a failure to recognize that parts of a fraction must be the same size. The second was when students were presented with number lines upon which fractions were placed with incorrect space. Students made this error, on average, 9% of the time on pre-quizzes and 10% of the time on post-quizzes. Twenty-nine percent of students made the illogical sizing/spacing error at least once on pre-quizzes and 35% of students

made it at least once on post-quizzes.

Table 5
Summary of Question Types and Their Errors

Question Type	Number of Questions	Errors Possible
Matching written fractions to visual models	8	complement illogical sizing/spacing incomplete information ratio reciprocal reducing fractions to whole numbers
Making equivalent fractions	4	complement incomplete information reciprocal reducing fractions to whole numbers
Placing fraction(s) on a number line	12	complement illogical sizing/spacing incomplete information reciprocal whole number ordering
Comparing fractions	6	same numerator/denominator ordering whole number ordering

Note. Question types are from the three fraction objectives.

3.1.4. Incomplete information. This error had two different characterizations.

The first was when the answer was partially correct but missing a critical part that would

make it correct (e.g., if there was a mark placed at one third of a number line but the

number line was zero to two, the student might have answered 1/3 when in reality the

answer was 2/3). The second was when the answer matched only part of the question.

For example, if the student was asked to identify the mixed number 1 3/4, a student

making this error would answer only 3/4 or only 1, missing the totality of the original

prompt. Students made this error, on average, 19% of the time on pre-quizzes and 8%

of the time on post-quizzes. Sixty-seven percent of students made the incomplete

information error at least once on pre-quizzes and 35% of students made it at least
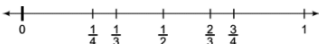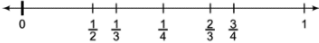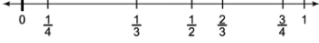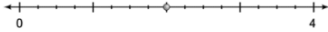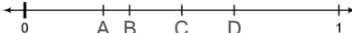
once on post-quizzes.

3.1.5. Ratio. Ratio errors were made when the student selected the fraction as the shaded parts over the unshaded parts instead of the shaded parts over all of the parts (Ashlock, 2001). Thus, if a circle was split into four equal parts with one of the parts shaded, a student making this error would select 1/3 instead of 1/4. On average, students made this error 7% of the time in pre-quizzes and 2% on post-quizzes. About 7% of students made the ratio error in pre-quizzes and 6% of students made this error in the post-quizzes.

3.1.6. Reciprocal. The reciprocal error occurred when the student flipped the numerator and denominator. For example, if the question asked what fraction represents the shaded part of a shape and the answer was 2/3, a student making the reciprocal error would select 3/2. Students made this error 15% of the time on pre-quizzes and 10% of the time on post-quizzes. Roughly half of the students made this error at least once on pre-quizzes (49%) and post-quizzes (45%).

3.1.7. Reducing fractions to whole numbers. Unlike whole numbers, fractions can be equivalent to both other fractions and to whole numbers (Bottge et al., 2014). This error occurred when a student did not correctly identify fractions that were equivalent to whole numbers. For matching written fractions to visual models, a student may answer that a circle with one section of five shaded is equivalent to the fraction 5/5. As for matching fractions, a student making this error might incorrectly state that three is equal to 3/3. Answer choices that corresponded to these errors all had fractions that could be reduced to a whole number. The fractions in these answer choices followed one of three forms: (1) a/a, where the fraction could be reduced to one; (2) 0/a, where the fraction could be reduced to zero; or (3) a/1, where the fraction could be reduced to

a. Students made the reducing fraction error 17% of the time on pre-quizzes and 10% of the time on post-quizzes. Approximately 66% of students made this error on the pre-quizzes and 44% made it on the post-quizzes.

Figure 1. Example of errors in the quizzes. The answer choices highlighted in each questions represents the labeled error. For the filler question, both C and D were coded as filler.

Table 6
Total Errors Made in Quizzes

	Mean	Standard Deviation	Minimum	Maximum
Pre-Quiz	6.621	2.757	0	13
Post-Quiz	4.135	2542	0	14

Table 7
Summary of Error Types

Error	Description	Proportion of Times Error is Made			Max Times Error Was Made	Number of Students Who Made This Error
		Mean	**Min**	**Max**		
Complement	Complement of correct fraction (e.g., correct: 2/3, incorrect 1/3)	0.11	0.00	1.00	3	421
		0.09	0.00	0.67	2	327
Filler	Incorrect answer but no obvious error	0.11	0.00	0.71	4	706
		0.07	0.00	0.50	3	843
Illogical size/spacing	Unequal parts of a shape or illogical spacing on the number line[†]	0.09	0.00	0.75	3	422
		0.10	0.00	0.75	3	503
Incomplete information	E.g., leaving out the whole number in a mixed fraction	0.19	0.00	1.00	4	855
		0.08	0.00	0.67	2	338
Ratio	The fraction is the shaded part over the unshaded parts[†]	0.07	0.00	1.00	1	99
		0.02	0.00	0.67	2	82
Reciprocal	Reciprocal of correct fraction (e.g., correct: 2/3, incorrect 3/2)	0.15	0.00	0.75	3	705
		0.10	0.00	0.60	3	639
Reduction to whole number	Misconception of what fractions are equivalent to whole numbers	0.17	0.00	0.80	4	938
		0.10	0.00	0.60	3	635
Same numerator/ denominator ordering error	Ordering incorrectly when the fractions had the same numerator or denominator	0.12	0.00	0.33	1	498
		0.08	0.00	0.50	2	399
Whole number ordering	Ordering only the numerator or denominators [†]	0.26	0.00	0.60	6	1,282
		0.12	0.00	0.57	4	822

Note. Statistics for pre-quiz errors are on top and statistics for post-quiz errors are below. N=1,431.
[†] a priori code.

Appendix A includes a table displaying correlations between error types.

3.1.8. Same numerator/denominator ordering error. This ordering error was possible only when students were asked to compare fractions with the same numerator or denominator in the comparing fraction questions. Within these questions, the student would only have to compare the numerator or denominator, but still made an error. For example, a student may say 1/3 > 2/3 (same denominator). There was one question in the pre-quiz and two questions in the post-quiz for which this error was possible. Students made the same numerator/denominator ordering error 12% of the time on pre-quizzes and 8% of the time on post-quizzes. Approximately 35% of students made this error on the pre-quizzes and 28% made it on the post-quizzes.

3.1.9. Whole number ordering. The whole number ordering error was made when the student ordered the fractions based on their numerator or the denominator, without apparent consideration of the relationship between the two (Malone & Fuchs, 2016). For example, a student might order 1/2, 1/3, and 1/4 in that order, because two is less than three and three is less than four. In this case, they are only ordering based on the denominators and not considering the fraction as one number defined by the numerator/denominator relationship. Students made the whole number ordering error, on average, 26% of the time on pre-quizzes and 12% of the time on post-quizzes. Almost all students made this error at least once on pre-quizzes (90%) and over half of students made it at least once on post-quizzes (57%). Students made the whole number ordering error the most in the pre-quiz and post-quiz.

3.2. Pre-Quiz Latent Class Analysis

As noted above, nine types of errors could be made in the third-grade quizzes— complement, filler, illogical size/spacing, incomplete information, ratio, reciprocal,

reducing to whole numbers, same numerator/denominator ordering, and whole number

ordering. Analyses were run in *Mplus* using the proportion of times each of the nine

errors were made to identify categorical latent variables.

       3.2.1. Fit criteria. A comparison of models with one to five classes is found in

Table 8. There was no model that displayed the best fit across every fit statistic.

Although the two-class model had the highest Bayes Factor, the two-class model did

not contain obvious patterns. Conversely, the four-class model had the lowest BIC but

did not have enough distinction between the classes. However, the three-class model

had the second lowest adjusted BIC value of the models (above 10; Kass & Raftery,

1995) and was the most interpretable; therefore, the three-class model was determined

to be the best-fitting.

Table 8
Table of Third Grade Pre-Quiz LCA Values

Class	Sample Size Adjusted BIC	BLRT p-value	Bayes Factor
1	21746.513		
2	21493.295	<0.001	>10
3	21430.938	<0.001	<1
4	21422.483	<0.001	<1
5	21454.988	<0.001	<1

Note. Entropy for three-class model is 0.543. The three-class model was chosen due to low sample size adjusted BIC and interpretability of the model.

       3.2.2. Pre-quiz error classes. Figure 2 shows the structures of fraction errors in

each class. The classes were examined holistically and named based on the prominent

pattern of errors. The naming pattern followed one of two options. The first is that the

class name is the same as the errors the students in that class made the most often.

The second naming pattern was used when there was no error was made significantly

more often than the other errors. In this case, the name represented the distribution of

all errors. The classes for the pre-quiz errors were the Distributed Errors Class, Whole

Number Ordering (WNO) Class, and Few Errors Class. Students were approximately

evenly distributed within the pre-quiz classes (30%, 35%, and 35%, respectively).

Students in the Distributed Errors Class had an average pre-quiz score of 44% (13sd) and an average post-quiz score of 65% (16sd).

Students in the WNO Error Class had an average pre-quiz score of 50% (12sd) and an average post-quiz score of 63% (14sd).

Students in the Few Errors Class had an average pre-quiz score of 71% (16sd) and an average post-quiz score of 86% (10sd).

☐ Did not make the error   ☐ 0-25%   ☐ 25-50%   ☐ 50-75%   ■ 75-100%

Figure 2. Pre-quiz latent class model. Each color represents how often students made the error. The y-axis represents how many students in that class made a specific error a certain percentage of the time (e.g., in the Few Errors Class, over 86% of the students did not make the complement error and about 12% made the complement error 25%-50% of the time).

3.2.2.1. Distributed Errors Class. The first class had the widest distribution of

errors, meaning that they made the most error types of all of the classes. The majority

of the errors were made at least once, with six of the nine errors being made at least

25% of the time. About 30% of the students (n = 432) were placed in the Distributed

Errors Class.

   3.2.2.2. Whole Number Ordering Error Class. The students in the WNO Class

primarily made the whole number ordering error. Although these students also made

other errors, the majority of students made the WNO error at least 25% of the time but

rarely made other errors this often. Approximately 40% of the students (n = 572) were

placed in the WNO Class.

   3.2.2.3. Few Errors Class. Students in this class primarily did not make errors,

except for the whole number ordering error. However, this class is distinct from the

WNO Class because errors were made so infrequently—over 70% of the students in

this class did not make six of the nine errors. Of the sample, 30% (n = 427) were placed

in the Few Errors Class.

   3.2.3. Predicting class membership. To determine who was in each class,

separate logistic regressions were conducted for each class (0/1 whether the student

was a member of that class). When running logistic regressions, an odds-ratio of

greater than one indicates an *increase* in likelihood of membership. Conversely, an

odds-ratio of less than one indicates a *decrease* in likelihood of membership. If an odds

ratio is less than one, taking the difference of the odds ratio and one can be interpreted

as the decrease in odds. Student pre-quiz emerged as a statistically significant predictor

of membership for each class. For each point increase in overall average pre-quiz

score, students had a 1.12-fold increase in the odds of being classified into the Few

Errors Class. However, for each point increase in average pre-quiz score, student odds

of being classified in the Distributed Errors Class decreased by 0.06-fold. Similarly, for

each point increase in average pre-quiz score, student odds of being classified in the

WNO Class decreased by 0.02-fold. Therefore, students who had higher pre-quiz

averages tended to be in the Few Errors class, whereas students who had lower pre-

quiz averages tended to be in the Distributed Errors or WNO Class. No other predictors

and demographics were statistically significantly related to class membership for either

the Few Error or the Distributed Errors Classes. However, if a student was classified as

gifted, the odds of them being in the WNO Class decreased by 0.51-fold.

Table 9
Logistic Regressions for Pre-Quiz Class Membership

	Distributed Errors		WNO Error		Few Errors	
	Odds-Ratio	Z-score	Odds-Ratio	Z-score	Odds-Ratio	Z-score
Male	1.24	1.65	0.80	-2.00	0.90	-0.70
Content						
Progress	1.00	-0.09	1.00	0.05	1.00	0.09
Average pre-quiz	**0.94**	**-15.38**	**0.98**	**-7.42**	**1.12**	**17.60**
Disability	0.87	-0.60	1.13	0.62	0.90	-0.41
Free Lunch	0.98	-0.13	1.24	1.53	0.68	-2.31
ELL	1.10	0.44	1.25	1.14	0.60	-1.57
Gifted	1.35	1.60	**0.49**	**-4.02**	1.37	1.40
Race						
Asian	1.33	1.08	0.82	0.75	0.84	-0.54
Black	1.13	0.59	1.26	1.39	0.70	-1.53
Hispanic	0.93	-0.35	1.02	0.13	1.09	0.38
Other	1.22	0.70	1.24	0.76	0.63	-1.37
Constant	**11.60**		1.95		0.00	

Note. Odds-ratios and Z-scores are provided. Reference groups: White, non-English Learner, not eligible for free/reduced lunch. Clustered by teacher. N=1,431.
Bold values are significant at *p* < .001.

3.3. Latent Transition Analysis

For the latent transition analysis, a post-quiz model was constrained to three

classes for two reasons. First, the three-class model had the lowest sample size

adjusted BIC value (see Appendix B). Second, it allowed the number of classes to

remain constant across the pre- and post-quiz models. The three-class post-quiz model
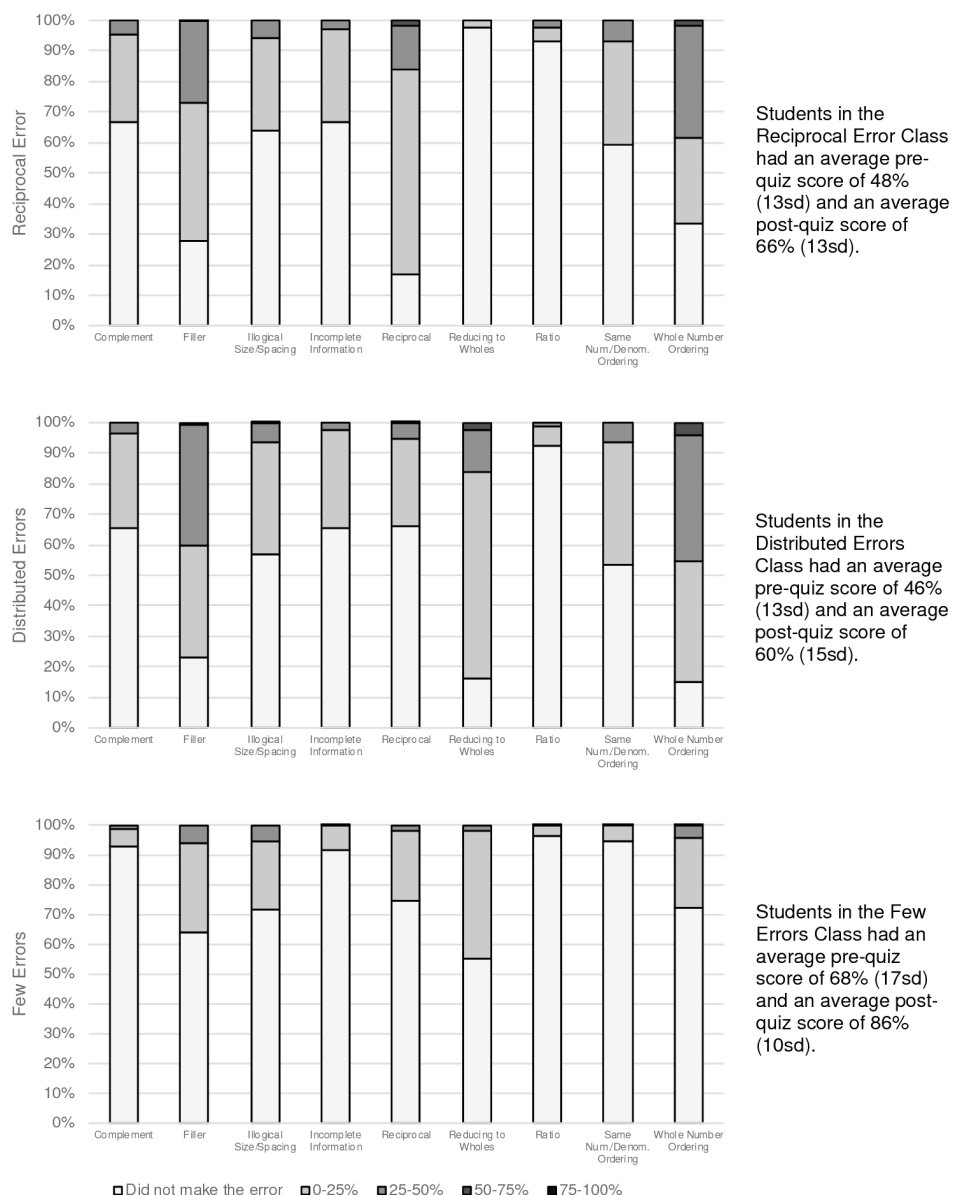
was estimated by regressing the post-quiz model onto the pre-quiz model. This regression determined the thresholds for the post-quiz classes. The thresholds determine how often each error is made in each class, i.e., what the prominent pattern of errors is for each class. Additionally, the regression determines the transition probabilities based on the models, i.e., how likely a student is to remain in a given class for pre-quiz and post-quiz.

3.3.1. Post-quiz model. Figure 3 shows the structures of fraction errors in each class for the post-quiz errors. As with the pre-quiz classes, colors indicate the percent of times, on average, students in that class made the particular error out of their total errors made and the y-axis represents how many students in that class made the particular error the specified proportion of time. In the same manner as was done for the pre-quiz Latent Class Analysis, the post-quiz classes were examined holistically and named based on the prominent pattern of errors. The post-quiz error classes were the Reciprocal Error Class, Distributed Errors Class, and Few Errors Class.

3.3.1.1. Reciprocal Error Class. Students in this class made errors that were largely distributed across the types, as is illustrated by the relatively high amount of blue in the bars (the error was made between 0% and 25% of the time). However, over 80% of the students made the reciprocal error at least once, in stark contrast to the percentage of students who made this error in the other two post-quiz classes. Approximately 30% of students (n = 437) were in the Reciprocal Error Class.

3.3.1.2. Distributed Errors Class. Similar to the Distributed Errors Class in the pre-quiz, the post-quiz Distributed Errors Class comprised students who made a variety of errors, but the reducing fraction to whole numbers and whole number ordering errors

were made most often. However, students also made other errors more often than in

the other classes. Of the sample, 29% (n = 415) were in this class.

Students in the
Reciprocal Error Class
had an average pre-
quiz score of 48%
(13sd) and an average
post-quiz score of
66% (13sd).

Students in the
Distributed Errors
Class had an average
pre-quiz score of 46%
(13sd) and an average
post-quiz score of
60% (15sd).

Students in the Few
Errors Class had an
average pre-quiz
score of 68% (17sd)
and an average post-
quiz score of 86%
(10sd).

Figure 3. Post-quiz latent class model. Each color represents how often students made the
error. The y-axis represents how many students in that class made a specific error a certain
percentage of the time (e.g., in the Reciprocal Class, over 90% of the students did not make the
complement error and about 8% made the complement error 0%-15% of the time).

3.3.1.3. Few Errors Class. Like the pre-quiz Few Errors Class, students in the

post-quiz Few Errors Class made the least errors. This class does differ from its pre-

quiz counterpart by having most errors made less often, especially in regards to the

relatively small number of whole number ordering errors. Approximately 40% of

students (n = 579) were in this post-quiz Few Errors Class.

      3.3.3. Differences in the classes. As with the pre-quiz, logistic regressions were

run to determine which demographic and game-play variables predicted class

membership. A student had higher odds (1.09-fold increase) of being placed in the Few

Errors Class if they had a higher pre-quiz average. Conversely, higher-performing

students were *less* likely (0.05-fold decrease) to be placed in the Distributed Errors

Class and the Reciprocal Error Class (0.03-fold decrease). Students were more likely to

be placed in the Few Errors Class if they were identified as gifted (2.14-fold increase)

but less likely to be placed in the Reciprocal Error Class (0.52-fold decrease). See

Table 10 for logistic regression statistics. Additionally, see Appendix C for average

proportion of times errors were made by each class.

Table 10
Logistic Regressions for Post-Quiz Class Membership

	Reciprocal Error		Distributed Errors		Few Errors	
	Odds-Ratio	Z-Score	Odds-Ratio	Z-Score	Odds-Ratio	Z-Score
Male	0.99	-0.05	0.90	-0.77	1.00	0.04
Content Progress	1.00	0.70	1.00	-1.37	1.00	0.90
Average pre-quiz	**0.97**	**-8.63**	**0.95**	**-12.16**	**1.09**	**15.87**
Disability	1.04	0.20	0.87	-0.65	1.06	0.24
Free Lunch	1.04	0.30	1.18	1.13	0.75	-1.78
ELL	1.56	2.14	1.02	0.08	0.53	-2.33
Gifted	**0.48**	**-3.74**	0.79	-1.20	**2.14**	**3.43**
Race						
Asian	0.92	-0.28	0.91	-0.33	1.15	0.48
Black	1.47	2.13	1.09	0.46	0.59	-2.49
Hispanic	0.93	-0.39	1.02	0.12	1.07	0.35
Other	0.91	-0.31	1.92	2.23	0.54	-1.90
Constant	1.60		**10.47**		**0.00**	

Note. Odds-ratios and Z-scores are provided. Reference groups: White, non-English Learner, not eligible for free/reduced lunch. Clustered by teacher. N=1,431.
Bold values are significant at p < .001.

3.3.4. Transition probabilities. Students who were in the Few Errors pre-quiz class, the class that was highest-performing at pre-quiz, moved only to the Reciprocal Error and Few Errors post-quiz classes, among which they were almost entirely placed into the post-quiz Few Errors Class (99.8%; only one of the 506 transitioned to the Reciprocal Error Class). Students who were in the pre-quiz Distributed Errors Class, the class that was lowest-performing at pre-quiz, moved to each of the three post-quiz classes. Primarily, these students were placed in the post-quiz Distributed Errors Class (56%), followed by the post-quiz Reciprocal Error Class (27%), with only 16% placed in the post-quiz Few Errors Class. Lastly, the students in the pre-quiz Whole Number Ordering Class, followed a similar transition pattern to those in the pre-quiz Distributed Errors Class. Of the WNO Class, 64% moved to the post-quiz Reciprocal Error Class, 35% moved to the Distributed Errors Class and 6% moved to the Few Errors Class. See Table 11 for a summary of the latent transition probabilities.

Table 11
Latent transition probabilities

| | | Post-Quiz Classes | | |
		Reciprocal Error	Distributed Errors	Few Errors
Pre-Quiz Classes	Distributed Errors	0.274	0.561	0.163
	WNO Error	0.642	0.352	0.006
	Few Errors	0.002	0.000	0.998

Note. Cells represent the probability that a student would be in the post-quiz class given that they were in the designated pre-quiz class. For example, if a student started in the Few Errors Class, there was a 0% chance they would be in the post-quiz Distributed Errors Class.

Logistic regressions were run to better understand which students transitioned between classes. Because all but one student who started in the pre-quiz Few Errors Class remained in the post-quiz Few Errors Class, logistic regressions were only run on students who started in either the Distributed Errors or WNO Class. Among these six transition possibilities, statistically significant predictors were only identified for

movement into the post-quiz Few Errors Class. If a student started in the pre-quiz Distributed Errors Class, the odds of them moving into the post-quiz Few Errors Class increased if they had a higher pre-quiz average or were classified as gifted (1.04-fold and 3.58-fold increase, respectively). Being classified as gifted also increased the odds of being placed in the post-quiz Few Errors Class if students started in the WNO Class (44.86-fold increase). However, being an English Language Learner also increased the odds of students transitioning from the WNO Class to the Few Errors Class (11.30-fold increase). See Appendix D for full logistical regression models.

4 Discussion

This study had three main questions—(1) What errors are made when students solve fraction problems? (2) How can students be classed by the errors they tend to make together? and (3) How do these classes change from pre-quiz to post-quiz?

4.1. Research Question 1: Fraction Errors in St Math

Within this study, we expand upon previous fraction error analysis in both the context and content of our questions and in the identification of our errors. First, breaking from the tradition of using open-response researcher-administered questions (e.g., Ashlock, 2001; Bottge et al., 2014; Brown & Quinn, 2006; Malone & Fuchs, 2016), we used a multiple-choice test provided in the course of students' actual instruction. Although the multiple-choice nature of our instrument limited the scope of errors students could make, it allowed us to examine errors in-situ within an authentic educational context. Additionally, if designed carefully and with common errors in mind, the use of such tests may make error identification more efficient—if researchers can make multiple choice answers with specific errors in mind, they can assume the error by

the answer choice instead of coding each written answer. However, because students

were required to pick an answer choice, selecting a random wrong (or right) answer was

more likely to occur. It is therefore possible that students may have picked an answer

choice that matched an error code without actually following the logic to get that error,

resulting in more noise. This may be especially true of students new to fraction

problems, who, on open-ended questions, might not provide any answer. By forcing an

answer choice, students who are new to fractions may make multiple errors. In fact, it

appeared that this may have borne out in the propensity of lower-performing students

within our pre-quiz Distributed Errors Class.

As our second contribution to the nature of error identification, we described new

categories of errors. Of the nine errors coded, six of them were a posteriori in that they

were not previously described in research on fraction errors—complement, filler,

incomplete information, reciprocal, reducing fractions to whole numbers, and same

numerator/denominator ordering error. These a posteriori errors were made just as

often, if not more, than those assigned a priori codes from prior research. It remains to

be seen if these errors would be found outside of the context of ST Math or in non-

multiple-choice examinations.

As the multiple-choice nature of the ST Math quiz questions may have influenced

the errors that arose, so too may have the visual nature of the questions. The focus

within ST Math on visual-spatial instruction is reflected in a number of quiz questions

that used visual representations, such as shaded figures or number lines. These

questions may elicit different errors than those seen in prior studies, which, other than

writing factions based on visual models, relied primarily on questions that were symbolic

(e.g., 1/2 + 3/4 = _____; Ashlock, 2001; Bottge et al., 2014; Brown & Quinn, 2006;

Malone & Fuchs, 2016).

4.2. Research Question 2: Models of Struggling Students

Three classes of errors were identified using LCA at pre-quiz—Few Errors,

Distributed Errors, and Whole Number Ordering Error. Students in the Few Errors Class

tended to have the highest pre-quiz averages. The students in this class made few

errors, and when they did make errors, tended to make the same error the majority of

the time. Although the LCA identified this group as a class, many of the students had

little in common regarding the type of errors they made—instead, they were joined

together merely by their propensity to make few errors. It is intuitive that high performers

would make few errors overall and that these errors may not be consistent among the

high performers. There are two possibilities we offer for why these students did not

make similar *types* of errors. First, it may be that even high-performing students

experience moments of carelessness, and therefore the distribution of errors these

students make is the result of a mostly random process. Alternately, it may be that there

are sub-groups within this class that cluster on their tendency to make certain types of

errors, but our instrument was not sensitive or extensive enough to capture these

differences.

The Whole Number Ordering Error Class contained less than five percent of

students who did not make the whole number ordering error and almost 60% who made

it at least half of the time. The high rate of this error—ordering only numerators or

denominators without considering the relationship between the two—is likely due to a

naïve understanding of fractions and an improper reliance on knowledge of whole

numbers and their ordering schema (Malone & Fuchs, 2016). Although other errors
were made by students in this class, none were as concentrated as the whole number
ordering error. Students in the WNO Class tended to score lower on the pre-quizzes
and were less likely to be labeled as gifted.

Lastly, the largest group of students were placed in the pre-quiz Distributed
Errors Class. This class made a wide range of errors but did not make most errors more
than 25% of the time. The seemingly random nature of their errors may indicate that
most of their errors are due to a general lack of fraction knowledge and unfamiliarity
with fraction problems. This conclusion is further supported by students in this class
having the lowest pre-quiz averages, indicating a lower level of fraction knowledge. This
may indicate that students with little fraction knowledge do not make errors that indicate
one type or a few types of conceptual or procedural misunderstandings, but instead,
exhibit a pattern that may be more indicative of guessing. It may be that these students
would be those that would leave open-ended questions blank, and therefore would not
be attributed to a specific error or classified into an error profile on the types of exams
frequently used in prior fraction error research.

4.3. Research Question 3: Class Membership Changes Pre to Post

Both class membership and class structure differed between pre- and post-
quizzes. This was expected, because the number of total errors being made decreased
as the students learned from ST Math. Two of the three pre-quiz classes remained in
the post-quiz—the Distributed Errors Class and the Few Errors Class. Interestingly, the
percent of students in the Few Errors Class increased by 10% from pre- to post-quiz.
The class also changed slightly in its composition of errors, as depicted in Figure 4,

wherein the pre-quiz error distribution is shown in the left columns and the post-quiz in

the right.
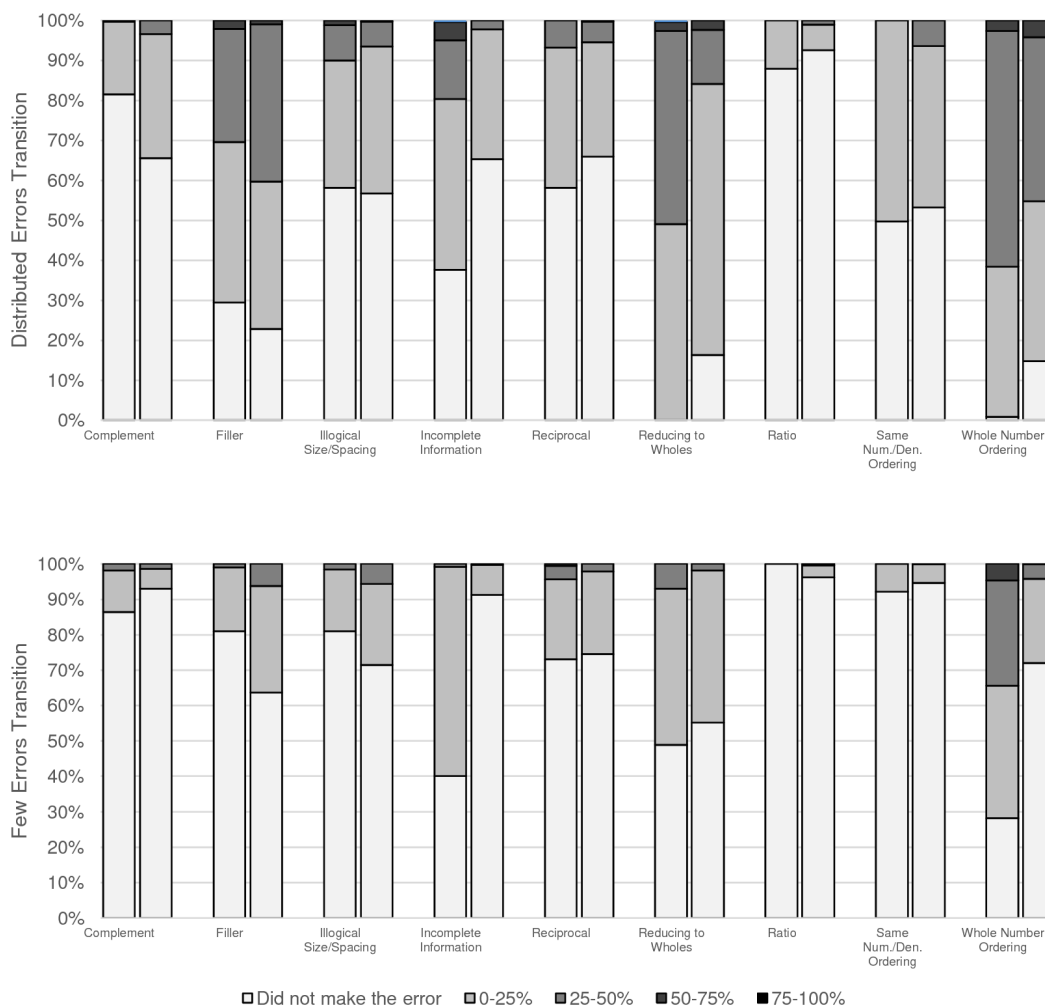
Figure 4. Distributed Errors and Few Errors classes transition. The left bars represent the pre-quiz class and the right bars represent the post-quiz class.

The most notable change from pre to post is that the proportion of errors

increased for the filler and illogical size/spacing categories in both classes and the

complement error in the Distributed Errors Class. For the illogical size/spacing error,

students who make this error have some aspect of the question correct—they

understand that 1/5 is one of five parts (for example) but do not recognize that the parts

must be equal, or they correctly order fractions but do not properly space them on the

number line. Thus, the students who make this error may have a partially developed

understanding of fractions. Alternately, students who selected answer options in line

with this error may have been confused at encountering what may be viewed as a *tricky*

option. Questions that elicit this error are necessarily complex–changing the spacing of

fractions on the number line may have taxed student working memory. The filler error

may have been more prevalent in the post-quiz classes because students were not

experiencing a specific misconception, rather they likely made careless errors. Thus,

although they have applied certain misconceptions in the past, those misconceptions

are suppressed and thus the student relies on a different strategy, in this case, choosing

an answer choice that does not fit under a specific misconception (see Booth et al.,

2014, describing the arithmetic error for an example of an error not representing a

misconception). This theory is supported by the Overlapping Waves theory that states

that children's strategies fluctuate as they learn (Siegler, 1996).

Only movement into the Few Errors class could be predicted by student

characteristics. Students who had higher pre-quiz averages and were labeled as gifted

were more likely to move from the Distributed Errors Class in the pre-quiz to the Few

Errors Class in the post-quiz. Gifted students, in addition to English Language Learners,

were also more likely to move from the Whole Number Ordering Class to the Few Errors

Class. Movement between the classes from pre- to post-quiz followed a pattern similar

to the one found with Resnick et al.'s (2016) fraction magnitude growth trajectories.

They found three growth classes—consistently accurate, inaccurate with growth, and

inaccurate with minimal growth. Although our project did not specifically look at growth,

we did find that students who made few errors in the pre-quiz continued to make few

errors in the post-quiz (similar to the consistently accurate class), students moved from

one of the lower performing classes—Distributed Error and WNO—to the post-quiz Few

Errors class (similar to the inaccurate with growth class), and some students were never

in either of the Few Errors class (similar to the inaccurate with minimal growth class).

4.4. Limitations

As noted, although the multiple-choice nature of the questions allowed us to

collect authentic educational data from a large number of students, it also provided a

limitation in that it constrained the possible number of errors to those already provided

within the software. It also allowed for the number of times an error could be made to

vary. For example, the same numerator/denominator ordering error could only be made

on one specific type of question—comparing fractions. Similarly, the number of times an

error could be made varied between the pre- and post-quiz, preventing a true measure

of difference, although this difference was small (differences ranging from zero to two

possible errors). Additionally, although we grounded our coding decisions in both prior

research of fraction errors and understanding of mathematics and mathematics

education, we lacked insight into each student's thought process as they solved the

problems. Future studies may rectify this shortcoming by providing for methods such as

cognitive interviewing to complement error coding.

4.5. Implications and Future Directions

Our results are immediately applicable to the digital platform from within which

the data came, ST Math. First, we can work with the platform developers to reduce the

number of filler items in ST Math quizzes and provide for a variety of error options with

which to identify student misconceptions. These same guiding principles can be used by

other test and platform developers to allow for more fine-grained data collection and identification of opportunities to assist student learning. Even after instruction, students demonstrated difficulty with the proper placement of numerator and denominator (reciprocal error). It may be that these are difficult concepts for all students, which would explain the persistence of these errors at post-quiz. Alternately, it could mean that ST Math is not presenting content in a way that teaches this idea as well as it does other fraction concepts. Examination of the curriculum and the puzzles that cover material related to these concepts can help to answer this question. Experimental studies altering this material and examining resulting error patterns can inform future iterations of both ST Math and other elementary fraction curricula.

Our work can also be applied to the realm of personalized or individualized instruction (e.g., learner profiles; Pane, Steiner, Baird, & Hamilton, 2015). As typically conceptualized, personalized learning aims to tailor each student's instructional needs to meet that student "where they are." This may be optimally accomplished by a computer program, such as ST Math, that can require mastery of a skill before a student moves on to more advanced skills. However, technology cannot work in isolation—knowledgeable teachers are necessary to support student learning (Anderson, Corbett, Koedinger, & Pelletier, 1995; Hew & Brush, 2007). By identifying typical patterns of student errors and grouping students based on the classes that define these patterns, teachers may be able to efficiently offer a type of personalized learning outside of the digital environment. At pre-test, a number of students displayed naïve schemas about fractions as demonstrated by the clustering of their errors around whole number ordering. The needs of these students in the classroom may be different than those who

have demonstrate what we view as a more general unfamiliarity with fraction problems

as seen in the Distributed Errors Class at pre- and post-quiz.

4.6. Conclusion

We set out to better understand third grade students' errors with fractions in a

digital learning environment using multiple choice tests of fraction concepts and

algorithms. We identified new types of errors, such as incomplete information and

reciprocal errors. We then examined the co-occurrence of these errors using Latent

Class Analysis, identifying three patterns of student errors at pre-quiz: one that

demonstrated a naïve fraction schema, one that contained high performers without

consistent errors, and one that included the majority of the students and likely reflected

student unfamiliarity with fraction problems. We found that, although some patterns

remained the same, such as distributed or few errors, some patterns changed after

instruction. Our work contributes to the growing field of research on fractions, adding to

the understanding of the complexity of fraction knowledge and offering insights into the

particulars of student struggles, insights that can contribute to the design of assessment

and instruction, and ultimately, to improved fraction and mathematics achievement.

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Appendix

Appendix A

| | | Pre-quiz | | | | | | | | | Post-quiz | | | | | | |
		1.	2.	3.	4.	5.	6.	7.	8.	9	10.	11.	12.	13.	14.	15.	16.	17.
Pre-quiz	1. Complement	1																
	2. Filler	.06[a]	1															
	3. Illogical Size/Spacing	.05	.15[c]	1														
	4. Incomplete Information	-.06	-.05[a]	-.06[a]	1													
	5. Reciprocal	.10[b]	.10[b]	.05[a]	-.03	1												
	6. Reduce to Whole Number	-.16[c]	.18[c]	.12[c]	.10[b]	-.12[c]	1											
	7. Ratio	.08[b]	.21[c]	.09[b]	.13[c]	>-.01	.07[b]	1										
	8. Same Num/Denom Ordering	.06[a]	.18[c]	.13[c]	.11[c]	.10[b]	.11[c]	.11[c]	1									
	9. Whole Number Ordering	.03	.06[a]	-.08[b]	.01	.12[c]	.16[c]	.02	.15[c]	1								
Post-quiz	10. Complement	.08[b]	.12[c]	.08[b]	<.01	.08[b]	.08[b]	.06[a]	.15[c]	.14[c]	1							
	11. Filler	.04	.17[c]	.12[c]	.08[b]	.11[c]	.15[c]	.06[a]	.16[c]	.27[c]	.08[b]	1						
	12. Illogical Size/Spacing	.05[a]	.10[b]	.09[b]	-.01	.04	.08[b]	.04	.08[b]	.16[c]	.08[b]	.14[c]	1					
	13. Incomplete Information	.09[b]	.17[c]	.08[b]	.08[b]	.09[b]	.09[b]	.05[a]	.10[b]	.14[c]	.08[b]	.13[c]	.01	1				
	14. Reciprocal	.06[b]	.11[c]	.03	.04	.22[c]	.02	.07[b]	.12[c]	.13[c]	.09[b]	.08[b]	.03	.06[a]	1			
	15. Reduce to Whole Number	>-.01	.08[b]	.07[b]	.03	-.01	.21[c]	.01	.07[a]	.09[b]	.01	.08[b]	.07[a]	.03	-.23[c]	1		
	16. Ratio	.03	.07[b]	.03	.01	.02	.02	.07[b]	.03	-.01	.03	.08[b]	.03	.07	.06	.05	1	
	17. Same Num/Denom Ordering	.07[b]	.19[c]	.11[c]	.03	.13[c]	.09[c]	.13[c]	.22[c]	.14[c]	.15[c]	.19[c]	.06[a]	.16[c]	.18[c]	.07[b]	.05	1
	18. Whole Number Ordering	.10[b]	.13[c]	.12[c]	.03	.13[c]	.12[c]	.10[b]	.21[c]	.27[c]	.21[c]	.27[c]	-.11[c]	.14[c]	.13[c]	.11[c]	.06[a]	.33[c]

Appendix B

Table of Third Grade Post-Quiz LCA Values

Class	Sample Size Adjusted BIC	BLRT *p*-value	Bayes Factor
1	20187.229		
2	19757.500	<0.001	>10
3	19693.396	<0.001	<1
4	19714.993	<0.001	<1
5	19741.642	<0.001	<1

Note. Entropy for three-class model is 0.664. The three class model was chosen due to low sample size adjusted BIC and interpretability of the model.

Appendix C

Table of Average Proportion of Times Errors Were Made by Each Class

	Pre-Quiz		
	Distributed Errors	Whole Number Ordering Error	Few Errors
Complement	0.05	0.19	0.08
Filler	0.17	0.12	0.04
Illogical size/spacing	0.15	0.07	0.05
Incomplete information	0.22	0.18	0.17
Reciprocal	0.10	0.23	0.10
Reduction to whole number	0.32	0.11	0.10
Ratio	0.13	0.07	0.01
Same numerator/ denominator ordering error	0.19	0.14	0.04
Whole number ordering	0.29	0.31	0.18
	Post-Quiz		
	Reciprocal Error	Distributed Errors	Few Errors
Complement	0.13	0.12	0.03
Filler	0.19	0.21	0.07
Illogical size/spacing	0.11	0.13	0.08
Incomplete information	0.13	0.12	0.02
Reciprocal	0.20	0.07	0.05
Reduction to whole number	0.00	0.22	0.09
Ratio	0.03	0.03	0.01
Same numerator/ denominator ordering error	0.13	0.14	0.01
Whole number ordering	0.17	0.22	0.04

Appendix D

Logistic Regressions for Movement out of Few Error Class

	Distributed Errors → Reciprocal Error		Distributed Errors → Distributed Errors		Distributed Errors → Few Errors	
	Odds-Ratio	Z-Score	Odds-Ratio	Z-Score	Odds-Ratio	Z-Score
Male	0.94	-0.27	0.95	-0.23	1.23	0.70
Content Progress Average	1.00	0.60	1.00	-1.86	1.00	1.55
pre-quiz	0.99	-1.12	0.99	-1.35	**1.04**	**3.10**
Disability	0.86	-0.37	0.81	-0.61	1.86	1.38
Free Lunch	0.56	-2.24	1.46	1.76	1.12	0.39
ELL	2.04	1.66	0.94	-0.18	0.29	-2.28
Gifted	0.47	-2.16	0.74	-1.05	**3.58**	**3.58**
Race						
Asian	0.44	-1.46	1.14	0.32	2.19	1.52
Black	1.27	0.68	1.32	0.85	0.33	-2.36
Hispanic	0.92	-0.23	1.01	0.03	1.08	0.19
Other	1.02	0.04	1.58	0.83	0.35	-1.25
Constant	0.52		6.13		0.00	

Logistic Regressions for Movement out of Distributed Error Class

	WNO Error → Reciprocal Error		WNO Error → Distributed Errors		WNO Error → Few Errors	
	Odds-Ratio	Z-Score	Odds-Ratio	Z-Score	Odds-Ratio	Z-Score
Male	1.36	1.50	0.75	-1.40	0.33	-0.65
Content Progress Average	1.00	0.21	1.00	-0.19	1.00	-0.73
pre-quiz	1.01	1.42	0.99	-1.46	1.01	0.23
Disability	1.10	0.30	0.91	-0.29	1.00	omitted
Free Lunch	1.15	0.61	0.88	-0.57	0.41	-1.23
ELL	1.17	0.47	0.81	-0.65	**11.30**	**3.49**
Gifted	0.80	-0.71	1.06	0.17	**44.86**	**5.79**
Race						
Asian	1.01	1.30	0.52	-1.22	1.00	omitted
Black	1.35	1.08	0.74	-1.07	1.00	omitted
Hispanic	0.85	-0.55	1.15	0.49	2.94	1.37
Other	0.58	-1.18	1.81	1.29	1.00	omitted
Constant	0.63		1.56		0.00	

Note. Odds-ratios and Z-scores are provided. Reference groups: White, non-English Learner, not eligible for free/reduced lunch. Clustered by teacher. N=1,431. Omitted variables predicted failure (i.e., students were not in the post class).
Bold values are significant at p < .001.