

Empirical Articles

Relational Equity and Mathematics Learning: Mutual Construction During Collaborative Problem Solving

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Abstract

We present an emerging interdisciplinary approach to the study of mathematics learning, which brings together strands from psychology and mathematics education. Our aim was to examine how students navigate the cognitive and social aspects of peer collaboration as they generate and adopt new strategies. We analyzed video data from a laboratory study involving pairs of elementary students working collaboratively to solve mathematical equivalence problems (e.g., $8 + 5 + 4 = 4 + \underline{\quad}$). We adopted a qualitative micro-analytic approach that focused on multimodal action (i.e., verbal utterance, gesture, inscription production, body positioning, and eye gaze) to examine three cases. These cases illustrate the complex ways that students interacted in this particular context and, in some instances, attempted to teach one another. Our findings show how “relational equity” (Boaler, 2008) and mathematics knowledge were co-constructed differently in each case. We argue that a micro-analytic approach, complemented by a blending of theory from these two fields, reveals hidden aspects of the interaction that may help explain, for example, why some students generate or adopt correct strategies and others do not. As such, this interdisciplinary approach offers a rich account of the learning processes that occur in peer collaboration.

Keywords: cognition, mathematics learning, mathematical equivalence, equal sign, relational equity, interaction analysis

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Motivation and Objectives

Researchers in psychology and mathematics education share a common interest in understanding the cognitive and social dimensions of learning. However, researchers from these two disciplines tend to draw on different methodological approaches that highlight different aspects of learning episodes. Research in psychology often relies on observations of students' problem-solving strategies in controlled laboratory settings, whereas research in mathematics education often relies on qualitative micro-analyses of instructional interactions. The study presented in this article seeks to strengthen bridges between these two fields by drawing on their respective methodological strengths and by synthesizing theory across them.

The micro-analytic approaches common in mathematics education research characterize and describe cognitive as well as social processes of learning at a fine-grained level of detail, which may support theoretical

insights that can explain variations in learning outcomes. Additionally, micro-analytic approaches can be leveraged to investigate questions related to equity in mathematics education, for example, questions about how social interactions relate to learning opportunities and outcomes in collaborative settings. Providing detailed expository accounts of students' mathematical reasoning vis-à-vis social interaction is not commonly the focus of laboratory-based psychology studies; as such, these micro-processes of learning may not be uncovered through traditional quantitative methods. That said, one strength of traditional psychological methods is the use of reliable instruments and standardized protocols. Laboratory studies allow for reliable measurement of knowledge states, strategy use, and learning outcomes, which are often subject to much theorizing but relatively little measurement in the micro-analytic approaches common in mathematics education.

The primary objective of this study was to investigate collaborative mathematics learning using an approach that blends theories and methods from psychology and mathematics education. We conducted a micro-analysis of peer interactions that took place in a laboratory setting, and in so doing, we sought to articulate a synthesized theoretical perspective on collaborative learning. This synthesized perspective takes into account mathematical and cognitive aspects of collaboration, while keeping a keen focus on issues of equity as they occur in moment-to-moment interactions between peers. In the sections that follow, we elaborate our emerging theoretical and methodological approach, present analyses of three cases involving elementary students engaged in collaborative mathematical problem solving, and describe implications for future interdisciplinary research that will increase our understanding of mathematics learning.

Theoretical Perspectives

This research investigates collaborative mathematics learning, with a particular focus on students' strategies for solving mathematical equivalence problems. In this section we first review relevant psychological literature pertaining to the development of strategies for solving equivalence problems. Next, we provide a brief overview of research in mathematics education as it relates to issues of equity in collaborative learning. Following these reviews, we outline a theoretical perspective focused on an adaptation of the idea of relational equity (Boaler, 2008), which guided our current study.

Understanding Mathematical Equivalence: A Psychological Perspective

Understanding that the two sides of an equation represent the same quantity is crucial for the development of later math skills and predicts success in algebraic reasoning (Fyfe, Matthews, Amsel, McEldoon, & McNeil, 2018; Knuth, Stephens, McNeil, & Alibali, 2006; Rittle-Johnson, Matthews, Taylor, & McEldoon, 2011). Algebra is often a *de facto* "gatekeeper" to advanced secondary mathematics courses as well as to higher education more broadly (Moses & Cobb, 2001; Oakes, Joseph, & Muir, 2004; Stein, Kaufman, Sherman, & Hillen, 2011), so promoting an early understanding of the equal sign as expressing a relation may have far-reaching social and economic consequences.

Despite the importance of understanding the equal sign, many students struggle to grasp that the equal sign expresses a relation. Instead, elementary students often view the equal sign in operational terms, that is, as a signal to "do something" (e.g., Matthews, Rittle-Johnson, McEldoon, & Taylor, 2012). When asked to define the equal sign, students commonly describe it as a signal to add all the numbers in the problem or to write down the answer (e.g., McNeil, 2007). Further, when elementary students solve mathematical equivalence problems

such as $8 + 5 + 4 = 4 + \underline{\quad}$, they often use strategies that do not treat the equal sign as a symbol calling for equal quantities on both sides (e.g., Perry, Church, & Goldin-Meadow, 1988). For example, students often add all of the numbers together (yielding 21) or add the numbers before the equal sign (yielding 17), rather than finding the value that would balance the equation (13; McNeil, 2007; Perry et al., 1988). Examples of strategies that students commonly use for solving mathematical equivalence problems are presented in Table 1.

Table 1

Common Strategies for Solving Mathematical Equivalence Problems. Based on Perry, Church, & Goldin-Meadow (1988)

Strategy name	Example written work	Example student utterance
Correct strategies		
Equalize	$8 + 5 + 4 = 4 + \underline{13}$	8 plus 5 plus 4 is 17 and 13 plus 4 is 17, so both sides are equal.
Add-subtract	$8 + 5 + 4 = 4 + \underline{13}$	8 plus 5 plus 4 equals 17, and 17 minus 4 is 13.
Grouping	$8 + 5 + 4 = 4 + \underline{13}$	There is a 4 on each side, so those cancel. 8 plus 5 is 13.
Incorrect strategies		
Add all	$8 + 5 + 4 = 4 + \underline{21}$	8 plus 5 is 13 plus 4 is 17 plus 4 is 21.
Add to equal	$8 + 5 + 4 = 4 + \underline{17}$	8 plus 5 plus 4 is 17.

A large body of research has focused on students' understanding of mathematical equivalence and potential methods to improve this understanding. This research has suggested that one key to solving equivalence problems correctly is accurately noticing the atypical location of the equal sign in the problems (Alibali, Phillips, & Fischer, 2009; Crooks & Alibali, 2013; McNeil & Alibali, 2004). When elementary students are asked to recreate equivalence problems from memory, they often recall the structure of the original problems inaccurately, and they sometimes produce problems with the equal sign at the end, in "operations equal answer" format (e.g., they sometimes reconstruct $8 + 5 + 4 = 4 + \underline{\quad}$ as $8 + 5 + 4 + 4 = \underline{\quad}$, Alibali et al., 2009; McNeil & Alibali, 2004). Accurate encoding of the position of the equal sign is associated with using correct strategies to solve the problems (e.g., Alibali, Crooks, & McNeil, 2017; Crooks & Alibali, 2013; McNeil & Alibali, 2004).

This association between accurate encoding of equivalence problems and the use of correct strategies is well documented. However, the processes through which accurate encoding actually *leads* to the generation and use of correct strategies have not been subject to systematic observation. The current study combines approaches from psychology and mathematics education to explore how students working together notice the equal sign, use that knowledge to construct correct strategies, and adopt or do not adopt these new strategies. Our focus on students working together raises many questions about social aspects of mathematics learning, to which we turn next.

Relational Equity From a Mathematics-Education Perspective

A growing body of scholarship on the social context of learning is concerned with equity in mathematics education (Diversity in Mathematics Education, 2007; R. Gutiérrez, 2013). Rochelle Gutiérrez (2012) presents a definition of equity in mathematics education that includes four dimensions: access, achievement, identity, and power. Briefly, *access* refers to "tangible resources" (R. Gutiérrez, 2012, p. 19) such as qualified and experienced mathematics teachers, high-quality curriculum, and reasonable class sizes, to name but a few. *Achievement*, then, relates to the outcomes that result from access to these resources, and these outcomes

are typically measured in terms of course enrollment patterns, performance on standardized exams, and broader participation in math-based fields. The dimension of *identity* refers to how students view themselves, each other, and their broader communities, in relation to mathematics. Lastly, *power* refers to advancing student voice and agency for both self-transformation and social change (see also R. Gutiérrez, 2002). According to Gutiérrez, access and achievement comprise the “dominant axis” of mathematics education, which has to do with the status quo of mathematics in society (i.e., “playing the game,” R. Gutiérrez, 2009), and identity and power make up the “critical axis,” which has to do with transformation and social change (i.e., “changing the game,” R. Gutiérrez, 2009).

Contemporary research in mathematics education focuses on creating and sustaining practices that support the various dimensions of equitable mathematics learning. For example, classroom-based research has highlighted the importance of participation structures and other curricular features for equitable mathematics learning (Boaler, 2008; Boaler & Staples, 2008; Esmonde, 2009; J. F. Gutiérrez, 2013; R. Gutiérrez, 2012; Hand, 2010; Langer-Osuna, 2011). These studies are part of a growing literature that addresses students’ content learning vis-à-vis critical issues related broadly to identity and power. These studies provide fine-grained analyses of interaction and reveal patterns of participation and positional identities as they emerge over short periods of time. Importantly, analyses of student interactions during moments of mathematical activity reveal variation in the ways in which students support each other to learn mathematics, which may ultimately influence long-term trajectories of identity and engagement (see, e.g., Langer-Osuna, 2011; Wood, 2013). However, these studies have an important limitation: they do not delve very deeply into the cognitive aspects of mathematics learning that occur during these social interactions. This article contributes to this body of research by examining students’ learning about mathematical equivalence and the social interactions in which new mathematical ideas emerge.

Most relevant to this research is the theory of “relational equity” (Boaler, 2008) which is defined as “equitable [social] relations in classrooms; relations that include students treating each other with respect and considering different viewpoints fairly” (p. 168). Boaler originally coined the term “relational equity” to describe equitable classroom relations that were supported and developed as a result of a specific instructional intervention called Complex Instruction (Cohen & Lotan, 1997), which was conceptualized as an equitable teaching approach. When implemented in mathematics classrooms, Complex Instruction involves a number of design features intended to promote equity, including heterogeneous, mixed-ability classes (i.e., no ability grouping); the use of open-ended, challenging problems that promote discussion and negotiation among all members of a group; and teacher moves such as “assigning competence” and other status interventions. Some studies that assessed Complex Instruction have reported high mathematics achievement among secondary students who learned via this approach (Boaler & Staples, 2008). Moreover, students in Complex Instruction schools were also reported to interact in ways characterized by relational equity, specifically along three dimensions: (1) *respect* for other people and their ideas; (2) *commitment* to the learning of others; and (3) adoption of learned methods of *communication* and support (Boaler, 2008).

In this work, we draw on Boaler’s framework to develop an analytic lens focused on *respect* for others and *commitment* to the learning of others, to complement our analysis of changes in mathematical understanding.ⁱ We argue that when students interact around mathematics, respect and commitment both shape and are shaped by micro-processes of collaborative learning.

The Current Study

In this paper, we leverage intensive, qualitative micro-analyses of multimodal behaviors (e.g., Abrahamson, Lee, Negrete, & Gutiérrez, 2014; Nathan, Wolfram, Srisurichan, Walkington, & Alibali, 2017; Nemirovsky, 2011; Schoenfeld, Smith, & Arcavi, 1991) to investigate how relational equity is manifested and how strategies for solving equivalence problems emerge during and after peer collaboration. This analysis brings together traditions and research methods from psychology and mathematics education to paint a richer, more complete picture of collaborative learning than that offered by either discipline alone. From psychology, we draw methods for measuring performance on the target math problems as well as for collecting data in a structured environment—in which multiple pairs of students experience the same tasks under controlled conditions—to examine learning and collaboration. From mathematics education, we draw methods for fine-grained analysis and interpretation of the social interactions that occur during these collaboration episodes. This interdisciplinary approach allows us to address questions about the importance of constructive and equitable social interactions in learning important mathematical content.

Research Questions

In this study, we seek to understand the processes involved in collaborative mathematics learning, including both cognitive and social processes. Therefore, we draw on psychological perspectives on knowledge change and on the relational equity framework in order to examine collaborative learning in a laboratory setting. We pose two primary research questions:

- How do new strategies emerge and how are they taken up by students as they collaboratively solve mathematical equivalence problems?
- What is the nature of relational equity in the moment-to-moment interactions of students engaged in mathematical collaboration? Specifically, we seek to characterize and describe how “respect” and “commitment” emerge during social–mathematical activities.

Method

This study is part of a larger project that investigates elementary school students’ collaborative learning of mathematics. The overarching aim of the project was to investigate the effects of collaboration on strategy change in students learning about mathematical equivalence. To test hypotheses about collaboration and strategy change, a controlled laboratory study using a pre-assessment/post-assessment design was conducted. Initial quantitative results appear in Brown and Alibali (2015).

Participants

Thirty-eight pairs of second, third, and fourth grade students were recruited to participate with parental consent. Thirty-eight students were recruited from a database of research participants maintained by the research laboratory. Each participant’s parents were invited to bring their child and one of the child’s friends of a similar age to the laboratory. All thirty-eight pairs were included in the initial analysis without respect to prior mathematical achievement.

We chose to have students work with friends, rather than with unfamiliar peers, because students typically work with classmates or familiar peers in classrooms and other informal education settings. We consider the implications of this methodological decision in the discussion section.

The complete set of 38 pairs (76 individual students) included students who were entering 2nd through 4th grade (age range 7;4 – 9;8). Parents were given the option to report gender, race, ethnicity, and school information for their children. Forty-two participants were female and 24 were male; the parents of 10 students did not report their gender. Of the 76 participants, 60 (79%) were Caucasian, 1 was African American, 1 was Asian, 1 was Hispanic/Latino, and 3 were of mixed race. The parents of 10 students did not report their race or ethnicity.

Procedure

Each pair of students participated in one 45-minute session in which they completed a pre-assessment, a collaboration episode, and a post-assessment.

Pre-Assessment

Participants worked individually to complete a battery of tasks designed to assess their knowledge of mathematical equivalence. This battery included two warm-up addition problems in the format commonly found in mainstream elementary mathematics curricula (e.g., $8 + 4 + 6 = \underline{\quad}$), four mathematical equivalence problems (addition problems with addends on both sides of the equal sign, and with one addend repeated on both sides, e.g., $7 + 6 + 9 = 9 + \underline{\quad}$), which are less common in such curricula, and a brief measure of students' conceptual knowledge of mathematical equivalence. The conceptual knowledge measure consisted of three tasks: (a) define the equal sign, (b) judge whether the equal sign belongs in a category with relational symbols (" $>$ " and " $<$ "), operation symbols (e.g., " $+$ " and " \times "), or numbers; and (c) sort equations (including ones of the forms " $8 = 8$ ", " $9 = 3 + 6$ ", and " $4 + 2 = 3 + 3$ ") into sets that "make sense" or "don't make sense."

Collaboration Episode

Following the individual pre-assessment, students in each pair worked together to solve two equivalence problems: $8 + 5 + 4 = 4 + \underline{\quad}$ and $9 + 7 + 5 = \underline{\quad} + 9$. They were given a single large sheet of butcher paper (which they could use as scratch paper) and markers along with the first problem presented on a separate, smaller sheet of paper. The students were asked to work together to solve the problem, and then to write their answers on their own answer sheets. After they solved the first problem, they were given the second problem and again asked to work together to solve it. They were given no additional instructions about how they should work together. The collaboration episodes were filmed for later transcription and analysis.

Post-Assessment

Students individually completed another battery of tasks, which included four equivalence problems similar to those in the pre-assessment, the same conceptual knowledge measure that was used in the pre-assessment (but with different items in the sorting tasks), and six transfer problems. The transfer problems were similar to the mathematical equivalence problems students had encountered up to this point in the study, but they had different structural features. Two did not include a repeated addend, meaning that the addend on the right side of the problem was not identical to any addend on the left side (e.g., $8 + 4 + 5 = 6 + \underline{\quad}$); two had the answer blank on the left side of the equation (e.g., $6 + 5 + \underline{\quad} = 6 + 10$), and two were multiplication problems (e.g., $4 \times 5 \times 2 = 4 \times \underline{\quad}$).

Taken together, the pre- and post-assessments were used to determine whether the students had generated or adopted any new strategies during the collaboration episode. In all phases of the study, each equivalence problem was followed by a prompt that asked participants to rate how confident they were in their answer on a scale from 1 to 5. Students' ratings of their confidence are not the focus of the present analysis, but will be referenced as we discuss the transcripts of the collaboration episodes.

Coding

One individual coded all 76 participants' strategies on the pre-assessment and post-assessment. Strategies were coded from participants' solutions using the coding scheme based on that described in Perry, Church, and Goldin-Meadow (1988) and summarized in Table 1. For a subset of the participants ($N = 20$), strategies were double coded to assess reliability. The coders agreed on whether or not strategies were correct or incorrect in 98% of cases, and they agreed on the exact strategy code in 95% of cases. The initial coder's strategy codes were used to categorize pairs into groups based on their performance at pre-assessment and post-assessment.

Each collaboration episode was also coded for whether at least one student in the pair noticed the location of the equal sign in the mathematical equivalence problems. We defined "noticing the location of the equal sign" as any verbal utterance, physical action, or gesture that indicated knowledge or awareness of the location of the equal sign. For example, students might explicitly comment on the location of the equal sign, read the problem aloud correctly, copy the problem correctly, or point at the equal sign. Two coders independently viewed the videos of all fifteen pairs in which neither student succeeded on the pre-assessment (i.e., both solved zero or just one problem correctly; see below), and coded whether at least one of the students noticed the equal sign during the collaboration. The coders agreed on 100% of pairs (15/15) on whether at least one student in the pair noticed the equal sign during the collaboration episode.

For the conceptual knowledge measure, students received one point for successful performance on each of the three tasks. On the equal sign definition task, students were deemed successful if their definition revealed a relational understanding of the equal sign (e.g., "It means the same"). Definitions for a subset of participants ($N = 20$) were double coded to assess reliability; agreement was 100%. On the symbol sorting task, students were deemed successful if they sorted the equal sign with other relational symbols rather than with operation symbols or numbers. On the equation sorting task, students were deemed successful if they sorted 9 or more of the 12 non-standard equations correctly (the criterion used in prior work; McNeil & Alibali, 2000). At each assessment point, scores for the three tasks were summed to yield a composite conceptual knowledge score (range 0-3).

Focal Cases

Selection of Cases

The 38 pairs were divided into seven main categories based on students' performance on the mathematical equivalence problems (not including the transfer problems) at pre-assessment and post-assessment. A student who solved zero or just one problem (out of 4) correctly on a given assessment was considered to be "non-equivalent" (NE) at that assessment, whereas a student who solved two or more of the problems correctly at a given assessment point was considered to be "equivalent" (EQ) (see Table 2, below).

Table 2

Seven Categories of Pair Relationships, According to Whether or not Students Acquired and Used Correct Strategies Between Pre- and Post-Assessment

Pre-Status to Post-Status	Description	Total
1. NE, NE → NE, NE	Correct strategies not acquired by either student	5
2. NE, NE → EQ, NE	Correct strategies acquired by one student	8
3. NE, NE → EQ, EQ	Correct strategies acquired by both students	2
4. EQ, NE → EQ, NE	Correct strategies used by one student at both assessments	4
5. EQ, NE → EQ, EQ	Correct strategies used by one student at both assessments and acquired by the other student	8
6. EQ, EQ → EQ, EQ	Correct strategies used by both students at both assessments	10
7. EQ, NE → NE, NE	Correct strategies lost by one student	1
Total		38

Note. NE = Non-equivalent; EQ = Equivalent (see text).

From the 38 pairs, we identified three focal pairs for analysis in the following way: First, we limited our scope to pairs in which *both* students were non-equivalent at pre-assessment (subcategories 1, 2, and 3 in Table 2; $N = 15$). This decision was based on our interest in understanding processes of strategy change, and specifically, generation of new, correct strategies. Second, we then limited this subsample to pairs in which at least one student noticed the equal sign during the collaboration episode ($N = 13$). Previous research has demonstrated that students who accurately encode the equal sign have a higher probability of solving the problems correctly (Crooks & Alibali, 2013; McNeil & Alibali, 2004). Narrowing the sample in this way allowed us to focus our analyses on pairs that had relatively similar likelihoods of generating a correct strategy.

These 13 pairs fell into three subcategories: pairs in which neither student used a correct strategy during the post-assessment ($N = 4$), pairs in which one student used a correct strategy during the post-assessment ($N = 7$), and pairs in which both students used a correct strategy during the post-assessment ($N = 2$). Because we were interested in possible differences in relational equity that might be associated with generating (vs. not generating) correct strategies, we selected one pair from each of these three categories. Each pair was thus representative of its subcategory in terms of performance at post-assessment. In our analyses, we leverage micro-analytic techniques to examine the collaboration episodes in these three cases.

In accordance with our theoretical perspective, we hypothesized that participants might include or exclude one another from the mathematical conversation in subtle or more direct ways. Therefore, in selecting focal pairs, we chose a set of three pairs that varied in the general tone of the interaction and in how well the students “got along” during the interaction, so as to maximize the chance of capturing variations in interaction that might be relevant to issues of relational equity. The fact that the students in each pair were friends might suggest that they would treat each other with respect and consideration, generally; however, in this mathematical context, this was not always the case.

This method yielded three focal pairs that were similar in terms of their pre-assessment knowledge (see Appendix), and similar in that within each pair, at least one student noticed the atypical location of the equal sign during the collaboration episode. The pairs differed in terms of their use of correct strategies at post-assessment and in terms of the general tone of the interactions. Note that we are not claiming that the collaborative process in these pairs is representative of that in the full sample, or even representative of that in

the subcategories from which they were drawn. We sought to examine whether there was variation among the pairs in their manifest relational equity and to explore how respect and commitment might be related to cognitive and mathematical processes.

Boaler's (2008) relational equity framework was originally based on using Complex Instruction as an equitable teaching approach. As noted in the literature review, a key element in Complex Instruction is to teach students how to work together in collaborative ways and another key element is to provide tasks that cannot be solved by individuals, so as to promote interdependence. In attempting to bridge multiple methodological and theoretical perspectives, our laboratory study did not include these elements; instead we sought to reliably measure the effects of collaboration on individual strategy use and learning outcomes. We argue that the relational equity framework is a valuable lens with which to analyze these episodes of peer collaboration for two main reasons: (a) the three focal cases display clear evidence of student learning (on the part of some of the participants), as indicated by changes in scores from pre- to post-assessment; and (b) the three pairs engaged in a diverse range of social as well mathematical sense-making activities, which enabled us to explore variations in patterns of peer interaction that manifest key constructs of relational equity, such as respect and commitment to others' learning.

Case Participants

As noted, we focus in this report on three pairs of students. Table 3 provides relevant demographic information reported by the parents of the six case participants. (Note that some parents chose not to provide all the requested information).

Table 3

Case Participants' Demographic Information

Characteristic	Pair 1		Pair 2		Pair 3	
	Elsa	Morgan	Dylan	Shawn	Jenny	Marie
Age (years; months)	7;6	8;3	8;0	7;7	8;8	7;10
Gender	Female	Female	Male	Male	Female	Female
Race	White	White	White	n.r.	White	White
Grade	2nd	2nd	2nd	n.r.	entering 3rd	entering 3rd

Note. n.r. = not reported by parents.

Analysis Techniques

We prepared detailed transcripts that included the students' speech, gestures, body posture, actions on the materials (e.g., use and movements of the paper, markers, etc.), eye gaze, tone of voice, and facial expressions. Each transcript was parsed into "lines," where each line was defined as a segment of continuous speech, or a gesture or action taken without speech (such as making a quizzical facial expression). Responses that included only brief utterances such as "yes" or "uhm" were also considered to be lines. Each segment of data presented in the "Analysis and Discussion" section below is labeled by the speaker and its line number from the original transcription. For an explanation of transcript conventions, see Table 4 in the section that follows. Once the transcripts were complete, two or more of the authors reviewed the transcript and video and worked together to come to consensus about whether the transcript included all the relevant information.

In addressing our research questions, we hypothesized that relational equity (i.e., respect and commitment) and mathematical knowledge (i.e., the generation of correct strategies) are simultaneously co-constructed through students' discourse and actions. Thus, we developed an analytic approach that allowed us to document, analyze, and interpret these reciprocal cognitive and social processes. This approach follows in a long tradition of qualitative micro-analyses of interactions around specific mathematical activities (Abrahamson et al., 2014; Nathan et al., 2017; Nemirovsky, 2011; Schoenfeld et al., 1991).

Working closely with the video footage and enhanced transcripts, we first analyzed each collaboration session in terms of students' mathematical thinking. We started by first noting whether students noticed the equal sign and how they interpreted it. We then inspected the transcripts for evidence of correct and incorrect strategies and interpretations of the equal sign, as reflected in students' discourse and actions (see Table 1, above).

Secondly, for each collaboration session, we assessed the quality of respect and commitment as it emerged in moments of multimodal interaction. We define respect as valuing and taking interest in the contributions of the other student, and we define commitment as showing concern for the other student's learning. These dimensions of the interaction were manifested in specific behaviors which we describe here in a general way, and which we illustrate more specifically in our analyses of the three focal cases.

We analyzed and described the quality of the *respect* that students accorded or demanded of each other. In particular, we sought evidence (from facial expressions, eye gaze, body posture, discourse, and other actions observed in the data) that students valued and took interest in the contributions of their peers. For example, appropriate turn taking, asking questions, and attentive listening were taken as signs of high respect, whereas interrupting or ignoring the other were taken as signs of low or limited respect. Respect could also be communicated non-verbally; turning towards one another and maintaining eye contact were taken as nonverbal signs of respect.

Similarly, we analyzed and described the nature of the *commitment* that students showed to one another's learning. This notion is based on students' shared responsibility for each other's learning,ⁱⁱ particularly in cases involving asymmetric knowledge. Commitment is manifested at moments when one student articulates a specific strategy or idea but the other student expresses disagreement or confusion. Students display high commitment to their partner's learning when they make efforts to explain a strategy or idea or when they seek to reach or maintain mutual understanding. For example, a student might check on their peer's understanding, elaborate or repeat an explanation, or bring up additional examples in an effort to help their peer understand. Students display low commitment to their partner's learning when they are dismissive of their partner's lack of understanding, or when they make no efforts to reach mutual understanding.

In our analysis of the three cases presented in this paper and in our ongoing analyses of the larger data corpus, we find that commitment and respect are often co-present. For example, a kindly worded rejection paired with an explanation meant to guide the other student towards an alternative strategy could be a sign of high respect plus commitment to the other's learning, but an immediate rejection without any explanation would be seen as expressing limited or low respect as well as no commitment to the other's learning. However, it is important to note that high respect and high commitment are not identical, and they do not always coincide. In fact, elsewhere we have documented a case of peer collaboration characterized by low respect but high commitment (J. F. Gutiérrez, Brown, Estep, & Alibali, 2017).

This analysis of cognitive and social processes was necessarily intensive and collaborative because it sought to develop a set of arguments that could explain our observations. We discussed the students' behaviors with an eye on the dimensions of relational equity that were salient in the interactions, as well as on aspects of their mathematical thinking that were manifest in the discourse. We fully illustrate our approach in the section below.

Analysis and Discussion of Three Cases

We present an analysis of three pairs that entered the collaboration under similar conditions but that had divergent paths of learning (see Table 2, previous section). The three pairs are similar in terms of their prior knowledge and skill, as indicated by their pre-assessment scores. Moreover, within each pair, one or both participants noticed the atypical location of the equal sign, and participants generated and discussed different strategies for solving the problems. Findings from previous literature suggest that under these conditions, students would be well positioned to generate new strategies and apply them to solve problems correctly (e.g., Alibali, Crooks, & McNeil, 2017). The students in the cases we present had different learning outcomes, as indicated by their post-assessment scores. In our analysis of each case, we consider cognitive and social aspects of the collaboration, based on our adaptation of the relational equity framework.

Table 4

Summary of Transcript Conventions

Symbol	Description
//	Marks beginning and end of overlapping utterances
long dash “—”	Abrupt halt
two dots “..” at the end of text (2 sec)	Very slight pause, less than a second 2-sec pause
repeated letters	E.g., “fiive,” marks lengthened syllable, each repeated letter equals one “beat”
<i>italics</i>	Marks stress
CAPITAL LETTERS	Increased volume
(??), ((this))	Unclear or inaudible reading, tentative reading
Bracketed notes	E.g., “[keeping her gaze down, she leans in to write “8” on big sheet],” marks other actions or voice quality

Case Studies

For each of the three cases, we first provide a brief overview to highlight certain points of the interaction, anticipating the transcription and line-by-line analysis that follows.

Pair 1: The Case of Elsa and Morgan: No Correct Strategies Adopted

Elsa and Morgan both solved zero (out of four) problems correctly at pre-assessment, and their collaboration did not result in either student generating new, correct strategies (i.e., they both solved zero equivalence problems correctly at post-assessment). Morgan's conceptual knowledge score actually *decreased* from pre-assessment to post-assessment (from 1 out of 3 to 0 out of 3), and Elsa's conceptual knowledge remained low (0 out of 3 at both pre- and post-assessment).

Problem 1 ($8 + 5 + 4 = 4 + \underline{\quad}$) — The collaboration session with Elsa and Morgan began just after the interviewer placed Problem 1 on the table and encouraged them to work together to solve the problem. At the outset, Elsa attempted to influence the flow of their work and to control the materials laid out for them. For example, in the first lines of the transcript (lines 1.1–1.4), Morgan moved the problem sheet closer to them, but Elsa immediately moved it to a different location. Morgan seemed to want to argue about the placement of the problem sheet at first (line 1.5–1.7), but she acquiesced and they moved on. This opening slice of the interaction mirrors the broader collaboration, in which Morgan put forth an idea and temporarily resisted Elsa’s opposing idea before she gave in and adopted Elsa’s strategy.

We argue that this social dynamic prevented their engaging with the problems in ways that might have led to one or both students generating a correct strategy. An opportunity to learn was missed, in spite of the fact that one of the students, Morgan, noticed the location of the equal sign (line 1.7) and began to develop some intuition about the structure of the problem (lines 1.13–1.19). We argue that her nascent, “proto” strategy could have become full-fledged, had it been explored further, were it not for Elsa’s limited respect for Morgan’s contribution (lines 1.20–1.25).

In turn, we argue that the students’ prior knowledge and engagement with the problem affected their opportunities for establishing relational equity, and we support this argument in our analysis, below. Note that in this segment, each of the students creates her own inscription on the big sheet of paper.

1.1	Morgan:	[moves problem closer toward them both]
1.2	Elsa:	Can—can you not put that [problem sheet] on the paper? [moves the problem to the middle of the table]
1.3	Morgan:	Yeah. [goes to grab the problem sheet, perhaps to readjust it]
1.4	Elsa:	[reaches across the table again, places her hand on the problem sheet, as if to suggest to Morgan that she needn’t move it] Or how about there on the paper? That’s good, on the paper.
1.5	Morgan:	[loudly] Wait! [looks at Elsa]
1.6	Elsa:	[looking at Morgan, after a brief moment, she makes a subtle hand gesture by rotating her hand from palm-down to palm-back position, as if to suggest to Morgan, “What is it—say something?” in a micro-moment of frustration]
1.7	Morgan:	[as she looks away from Elsa, shifting her gaze down at the problem, and gesturing with a palm-up motion] Okaayy. [She indicates each number on the sheet with her marker as she speaks] Eight plus five plus four equals ff—and that’s the middle [indicates the equal sign], but four [indicates right 4] plus [indicating the blank] five plus eight won’t FIT. [looks up at Elsa, anticipating a response]
1.8	Elsa:	[keeps her gaze on the problem and vaguely gestures to it, perhaps indicating the blank] No, what would be after that?
1.9	Morgan:	[returns her gaze to the problem, then back up to Elsa; shrugs her shoulders slightly and talks with a tone that suggests she is uncertain] A fiive?
1.10	Elsa:	[briefly glances in the interviewer’s direction, then back to the worksheet] Wai—ooh yeah! Would—it’s a pattern! [quickly shifts her gaze to Morgan, then back to the problem]
1.11	Morgan:	Ayyy, yyeah! [exchanges glances with Elsa, then quickly shifts her gaze down at problem sheet]
1.12	Elsa:	And we have to figure it out. [exchanges glances with Morgan again] It’s eight, five, four [writes “8 5 4” on the big sheet].. ooh [looks up at Morgan] three! I think.
1.13	Morgan:	Wait, no! //Eight [keeping her gaze down, she leans in to write “8” on big sheet]..
1.14	Elsa:	//I think it’s three.//
1.15	Morgan:	Five [continues writing, “8 5”]..wait!
1.16	Elsa:	[lifts out of her seat, leans in a little more toward Morgan and the problem] Four.
1.17	Morgan:	Plus [adds plus sign to her inscription, “8 + 5”],

- //plus four [continues writing, "8 + 5 + 4"]..
- 1.18 Elsa: // [makes a sharp up and down movement with her left hand as she talks, with a tone suggesting she is frustrated] It's counting down.//
[adjusts in her seat, leans in farther, shifts her gaze between Morgan and the problem] It's counting down!
- 1.19 Morgan: Equals four plus five plus eight. [completes her inscription as she talks, "8 + 5 + 4 = 4 + 5 + 8"]
- 1.20 Elsa: [watched what Morgan was inscribing on the big sheet of paper; keeps her gaze down as she talks] No we have to, we have to—the answer to the problem is [adds 3 to end of her string of numbers, as "8 5 4 3," then looks at Morgan] three.
- 1.21 Morgan: [gazes at Elsa's string of numbers "8 5 4 3"] Oohh! [scribbles over her equation]
- 1.22 Elsa: BECAUSE it's counting down. [leans in, perhaps trying to make eye contact with Morgan as she scribbles; she smiles, puts marker down, picks up pencil, and writes "3" on her answer sheet and circles the middle rating on the confidence rating scale]
- 1.23 Morgan: Ooohh! Eight, five, four, three. [writes "8 5 4 3" above her scribble, then puts marker down, picks up pencil, writes "3" on her answer sheet and circles the middle confidence rating]
- 1.24 Elsa: Okay. [puts pencil down, places hands on desk, then briefly looks over in interviewer's direction as if to suggest that they are done with the problem]
- 1.25 Morgan: [glances down in Elsa's direction, possibly looking at her worksheet, or her hands, or both; she too places hands on desk and looks in the interviewer's direction]

Analysis and Discussion of Pair 1, Problem 1 — An analysis of the transcript above reveals at least two important components that bear on our main thesis regarding the mutual construction of relational equity and mathematics learning. First, we consider the students' mathematical reasoning. Morgan noticed and referred to the equal sign, but Elsa persuaded Morgan to abandon her emerging strategy in favor of Elsa's pattern strategy, which led to a mathematically incorrect statement. Second, this transcript reflects a dynamic social interaction. Morgan's and Elsa's mathematical contributions not only *indicate* their understanding of the problem, they also simultaneously shape and establish their relational equity. It is in this sense, we argue, that mathematical knowledge and relational equity were simultaneously co-constructed through their discourse and actions. The two dimensions of relational equity—respect and commitment—were manifest inside the personal and, importantly, interpersonal resources that students brought to bear in this problem-solving context. To unpack these two analytic components, we consider the interaction more closely.

Morgan began by reading the problem aloud, and she paused to highlight the location of the equal sign (line 1.7). Her comment, "that's the middle," reflects that she noticed its location in the problem, and may also reflect her knowledge of the equal sign as dividing the equation into two sides. However, in this same turn, she indicated that she did not know how to proceed because the answer space was small and did not allow for her proposed strategy of simply repeating all three addends on the right side of the equation (line 1.7). The fact that Morgan wanted to balance both sides of the equation reveals her nascent understanding of the equal sign as a symbol expressing a relation. Unfortunately, Morgan's insight was lost through the social interaction, from one turn to the next.

Elsa responded to Morgan's contribution with limited respect. She did not acknowledge Morgan's contribution and immediately suggested an alternative strategy (line 1.8). Elsa's proposed strategy did not treat the problem as an arithmetic problem; instead, she suggested that the problem was a pattern and that she and Morgan needed to find the number that would continue the sequence (lines 1.10, 1.12). Morgan briefly entertained this possibility as indicated by her verbal "yyeah" (line 1.11) and by starting to write the numbers in the problem without any operator symbols (lines 1.13, 1.15). Next, however, she seemed to realize that the symbols were key to solving the problem, because she went back and filled in the plus sign that she had skipped. She

correctly copied the problem and inserted her proposed solution for balancing the equation, so that her inscription read $8 + 5 + 4 = 4 + 5 + 8$ (lines 1.17,1.19).

At this point in the interaction, Morgan had exhibited two behaviors that are crucial for understanding the equal sign as a symbol expressing a relation. She had noticed the location of the equal sign, and she had articulated a strategy based on balancing both sides of the equation. Morgan had even written down a correct mathematical statement on the shared large paper (Figure 1b). However, there was also a competing strategy at play: Elsa's pattern-based strategy. When Elsa declared that the answer was three (line 1.20), Morgan quickly accepted this and abandoned her own emerging strategy by scribbling over her correct work, entirely obscuring it (Figure 1c). Morgan then re-wrote Elsa's work on her own paper and filled in her answer: "3."

Looking at the nature of the respect exhibited by the students, the dynamic evolved such that Elsa led the way and Morgan's ideas were marginalized. Elsa's interactions with Morgan served to control the space and the materials (lines 1.2, 1.4), assert her own ideas (lines 1.6, 1.8, 1.10, 1.12, 1.14, 1.16, 1.18, 1.20, 1.22), and reject Morgan's contributions (at line 1.20, for example). Elsa's interactions with Morgan do not give any indication that she valued or took an interest in Morgan's ideas. When Elsa made statements about a proposed strategy, she was asserting that she had the correct approach and Morgan did not. Elsa was not presenting another tool for Morgan's toolbox or inviting open discussion (moves that would show commitment to Morgan's learning); instead, her moves suggest that she intended to take and *keep* the floor. In these ways, the relational equity that was co-constructed during their multimodal interactions was limited in terms of respect and commitment to each other's learning. These interactional dynamics continued in Problem 2, which we examine next.

Problem 2 ($9 + 7 + 5 = \underline{\quad} + 9$) — On the second problem, Morgan attempted to establish herself as an equal member of the collaboration (line 1.29). Morgan suggested a strategy that is similar to—and likely derived from—the non-arithmetic, pattern-based strategy Elsa generated for Problem 1. Despite her best efforts, however, Morgan's ideas were again rejected (line 1.30), and the quality of relational equity that had been co-constructed in the previous problem carried over and remained, from a pedagogical perspective, less than desirable.

1.26	Elsa:	Okay, let's see..
1.27	Morgan:	Mmm..
1.28	Elsa:	[looking down at the problem] Nine... uhh. [giggling] I have no idea ((what to do)). [glances up at Morgan] What's nine plus seven plus—
1.29	Morgan:	[shifts her gaze back and forth, from Elsa to the problem] Wait! This is question number two. Since the other one was going <i>down</i> [makes a vague gesture pointing down with the marker in her hand], this one might be going <i>up</i> [makes a similar vague gesture, pointing the marker up].
1.30	Elsa:	[looking at the problem] Noo, because see [indicating numerals on the problem sheet] nine, seven—[quickly shifts gaze to Morgan] (??) counting by twos? No, counting by <i>odd</i> numbers.
1.31	Morgan:	Yeah!
1.32	Elsa:	What would be the next odd number? Ohm, one! Wait, uhm.. three! Three. [puts marker down and picks up pencil to write the answer, but then she pauses] Again? [glances at Morgan] I've got a... [indicates something on her answer sheet] yeah. Let's just do that as an answer. [writes "3" on her answer sheet]. Aii..
1.33	Morgan:	[she too writes "3" as final answer and circles the middle confidence rating; glances down at Elsa who is still writing on her sheet]
1.34	Elsa:	[circles the second-lowest confidence rating] Okay. [puts her pencil down]

1.35 Morgan: [after watching Elsa mark her confidence rating, changes her confidence rating from middle to second-lowest to match Elsa]

Analysis and Discussion of Pair 1, Problem 2 — The dynamics established during the first problem set the stage for this second problem. However, this time it was Elsa who initially suggested an arithmetic strategy (line 1.28), one that used the plus signs in the problem, which she had completely ignored in Problem 1. Once reminded, Elsa quickly returned to her pattern approach, but she also quickly rejected Morgan’s suggestion that the pattern “might be going up” (lines 1.29, 1.30), which again manifests limited respect for Morgan’s contributions. In the end, both students fully adopted Elsa’s idea that the goal was to find a pattern within the sequence of numbers, ignoring the plus and equal signs. Elsa continued to drive the interaction and proposed a final answer of “3”—the next odd number, counting down—that Morgan immediately took up (lines 1.30–1.35; see Figure 1d).

Given that an emerging correct strategy was thoroughly rejected on Problem 1 and not even discussed on Problem 2, perhaps it is not surprising that neither student used a correct strategy on the post-assessment. Morgan did solve one of the transfer items successfully, but it seems likely that she arrived at that particular correct solution using the (incorrect) pattern strategy that she adopted during the collaboration episode. Ultimately, this was a collaboration in which neither student benefited mathematically.

We summarize the co-construction of relational equity for the case of Elsa and Morgan as follows: Morgan noticed the location of the equal sign, which led her to articulate correct intuitions about the goal of the problem, yet this emerging strategy was never taken up or adopted by either student. Instead, the fact that Elsa controlled (and Morgan allowed her to control) the interaction meant that Elsa’s mathematically less-accurate strategy was adopted in place of Morgan’s more-accurate one. In sum, Elsa and Morgan’s collective actions manifested limited or “weak” forms of the respect and commitment that characterize relational equity.

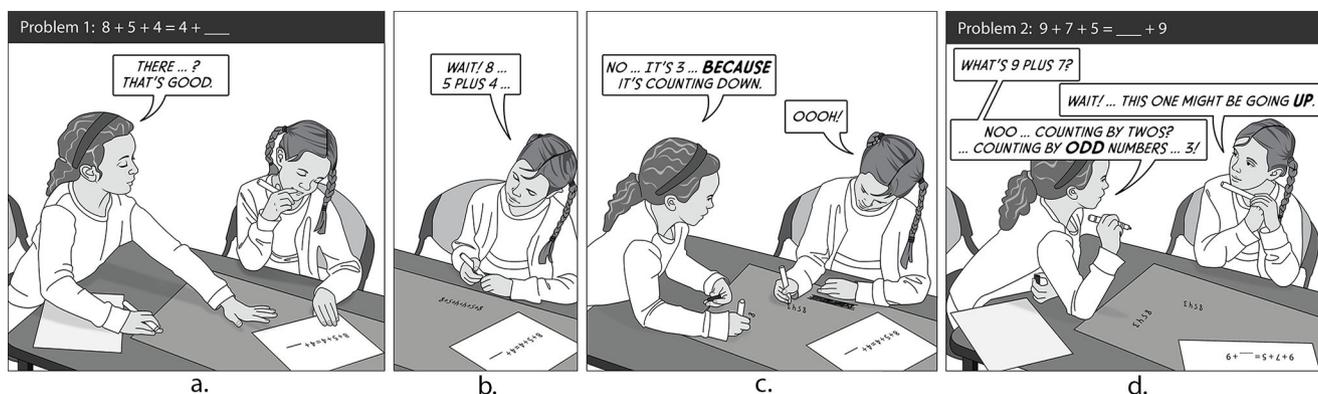


Figure 1. Synoptic comic strip of Elsa (left) and Morgan (right) working on Problems 1 and 2. From Alibali, Gutiérrez, & Brown (2017), under a CC-BY4.0 license.

Pair 2: The Case of Dylan and Shawn: Correct Strategies Generated and Adopted by One Student

In this session, both students noticed the location of the equal sign and both adopted a correct strategy to correctly solve both collaboration problems. However, despite both students taking up this strategy during collaboration, only one student, Shawn, used it consistently at post-assessment. Neither student improved on the conceptual knowledge measure (Dylan’s score remained at 0 and Shawn’s at 1 from pre-assessment to

post-assessment, out of 3 possible). We argue that this modest and asymmetric learning is associated with a lack of relational equity.

Problem 1 ($8 + 5 + 4 = 4 + \underline{\quad}$) — In this excerpt, Dylan began by adding the left side of the equation (lines 2.3, 2.5, 2.7). When he encountered the equal sign, he expressed uncertainty about what to do regarding the right side of the equation, and looked to Shawn for help. However, Shawn remained focused on the problem, and did not appear to consider Dylan's ideas. They essentially worked in parallel, with each constructing his own understanding of the problem. Shawn eventually generated a correct strategy (line 2.8), which provided Dylan the opportunity to adopt Shawn's approach, and he abandoned his own strategy (line 2.9). The only time Shawn looked away from the problem and in Dylan's direction was when he put forward his proposed solution (2.12).

2.1	Dylan:	[stands up from his seat; glances at Shawn, then at the problem]
2.2	Shawn:	[keeps his gaze down at problem] Eight plus five is..
2.3	Dylan:	Well let—let's go over to here [indicates left 4 using his finger, then quickly slides his finger to the right, gesturing toward the equal sign] 'cause I don't actually—hmm.
2.4	Shawn:	[now out of his seat, leaning forward over the problem] So eight plus five..
2.5	Dylan:	[still looking at problem] It's plus five and...
2.6	Shawn:	[gestures with the tip of his pencil toward the left 5 + 4] So nine plus //[indicates toward the 8] eigh—
2.7	Dylan:	//[indicating the numerals, the plus sign, and the equal sign as he reads the problem] I mean plus four// equals four plus [vaguely gestures to blank]—
2.8	Shawn:	[indicates left 8 with the tip of his pencil; speaking loudly over Dylan] So THAT's seventeen. [leaves tip of pencil on left 8; vaguely glances in Dylan's direction] Four plus <i>what</i> equals seventeen?
2.9	Dylan:	Four plus [shifts gaze to Shawn] whaat? Four pluuss [looks up at nothing in particular]... well //[looks at Shawn] it's four so of course there's ten.
2.10	Shawn:	//Four:// [quickly glances up at Dylan] Four plus three.
2.11	Dylan:	Wait! What?! [lifts gaze up, staring out again at nothing in particular]
2.12	Shawn:	[glances up at Dylan] Four plus thirteen! [reaches for problem, orients it so that he can write on it]
2.13	Dylan:	Four plus—oh yeah! [looks down and leans in over the problem] Yeah! Plus thirteen. [goes to write something but sees that Shawn is already writing something]
2.14	Shawn:	[writes 13 in the blank, writes 13 on his answer sheet, circles the highest confidence rating, then glances over at Dylan's answer sheet]
2.15	Dylan:	[writes 13 on his answer sheet; circles the highest confidence rating, looks up into space then back to his paper, then quickly erases it and circles the second-highest rating instead]

Analysis and Discussion of Pair 2, Problem 1 — At the outset of the collaboration session, Shawn immediately began adding the numbers on the left side of the equation, while Dylan gestured to the middle of the equation, suggesting that he was encoding the position of the equal sign. When Shawn proposed a strategy based on a correct understanding of the equal sign (line 2.8), Dylan abandoned his train of thought and attempted to follow Shawn's (line 2.9). Dylan's responses (lines 2.13, 2.15) suggest that he agreed with Shawn's strategy. However, he may have simply been following Shawn's method without understanding that the quantities on both sides of the equation must be the same (line 2.13). When Dylan recorded his answer (line 2.15), his facial expressions and posture suggest that he was still uncertain, despite the fact that he acquiesced to Shawn. It is noteworthy that he even revised his confidence rating downward at the end of this excerpt. Dylan's lack of confidence suggests he accepted Shawn's strategy without fully comprehending it.

In terms of relational equity, at first glance it appears that Dylan and Shawn were working together effectively. On the surface, this appears to have been a successful collaboration with equitable participation from both students. They reached agreement quickly and answered the question correctly. However, a close analysis of the interaction reveals that Dylan's enthusiasm and agreeable nature belied his actual mathematical understanding, and he was not given (nor did he demand) an opportunity to develop this understanding. Shawn made statements that were independent of any input Dylan was providing; for example, Shawn's suggestion in line 2.8 did not build on Dylan's contribution in line 2.7, suggesting that Shawn did not show a great respect for Dylan's contributions. In fact, Shawn seemed at times to talk *at* Dylan, not *with* him. Despite the fact that the students were cooperative, came to an agreement, and answered the problem correctly, these micro-behaviors in the interaction, similar to the case of Elsa and Morgan, suggest that the respect and commitment that characterize relational equity were expressed only to a limited extent.

In Problem 2, below, their apparent cooperation dissolved as Shawn and Dylan continued to work in isolation and talked past each other. Dylan attempted to assert himself as a participant in the conversation, to follow along with Shawn, and to develop his understanding of the problem. Yet his enthusiasm ultimately faded away as his learning was stymied by their interaction.

Problem 2 ($9 + 7 + 5 = \underline{\quad} + 9$) — When Dylan and Shawn first began working on Problem 2, their interaction showed promise of reflecting relational equity. They worked in tandem for several turns, and they successfully determined that the left side of the equation sums to 21 (lines 2.16–2.21). However, in approaching the right side, their communication broke down and they were no longer on the same page. Shawn seamlessly applied the correct strategy he had generated for Problem 1 (line 2.22); however, Dylan slid backward and proposed an incorrect strategy of adding all the numbers (line 2.23). This divergence resulted in a communication breakdown that went unrepaired. For the remainder of this interaction, Dylan attempted to make sense of what Shawn had proposed, but to no avail. The pair eventually offered the final answer of 12, which was proposed by Shawn. Despite his acquiescence, Dylan seemed unconvinced and even dejected. Although both boys wrote “12” on their worksheets, Dylan acknowledged that he thought the answer might be something different.

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|------|--------|---|
| 2.16 | Shawn: | [gets out of his seat and leans over the problem] Nine plus seven. [indicates the problem using the tip of his pencil] |
| 2.17 | Dylan: | So that would be, that would be... sixteen [glances at Shawn as he indicates the problem] |
| 2.18 | Shawn: | Sixteen [indicates the left 9] uhmm [then places tip of pencil on second plus on the left side]— |
| 2.19 | Dylan: | And sixteen plus five equals twentyyyy-onee [leans away from Shawn yet keeps his gaze on him] |
| 2.20 | Shawn: | Twenty-one— |
| 2.21 | Dylan: | Yeah! And then.. |
| 2.22 | Shawn: | Equals what..
//what plus [shifts gaze back and forth between problem and Dylan] nine equals twenty— |
| 2.23 | Dylan: | //Oh! That would be.. //
that would be twenty-one [indicates the blank] and then plus nine [indicates right + 9]. So.. hmm. |
| 2.24 | Shawn: | No, twelve! Twelve. [looks up at Dylan] |
| 2.25 | Dylan: | Wait, what?! [leans in to look at the problem] |
| 2.26 | Shawn: | Plus nine is twenty-one. [Looks up at Dylan] 'Cause, [stands up out of his chair, with a tone that suggests uncertainty] I think. |
| 2.27 | Dylan: | [looks at Shawn, shrugs, then leans to write the answer but then draws back] Wait, what? [looks at the problem and vaguely gestures toward it with his hand] Well, seven— |
| 2.28 | Shawn: | [interjecting, excitedly] Like two plus nine is ele— |

- //is eleven, right?
- 2.29 Dylan: //That one is nine plus seven. [indicates the left 9 and left 7; turns to face Shawn]//
- 2.30 Shawn: [facing Dylan] Two plus nine is eleven, right?
- 2.31 Dylan: [now squarely facing Shawn] Two plus nine?
- 2.32 Shawn: Yeah.
- 2.33 Dylan: No, that's—ohh yeah, that's eleven [gestures at Shawn, then turns to the problem].
- 2.34 Shawn: [looking at and indicating the problem] Yeah so—
- 2.35 Dylan: That's—that's seven [quickly indicates left 9, 7, and 5 as he speaks]—that's
//nine plus seven..
- 2.36 Shawn: //Yeah so//
twenty tah—twenty-one, that's twenty-one [points at the left side with his pencil] equals
//twelve [points at the blank] plus nine, right?
- 2.37 Dylan: ///[shrugs] I guess.///
[5 second pause; stares away, out into space, suggesting that he is taking a moment to consider].
- 2.38 Shawn: [calmly watches Dylan]
- 2.39 Dylan: [smiles then lifts up a finger and points to Shawn, suggesting that he still needs a moment] Waaait.. [breaks his concentration, laughs sheepishly]
- 2.40 Shawn: [looks away and laughs a little]
- 2.41 Dylan: [stares down at the problem; 4 second pause] Wait.. what? [his hands drop to the table and his shoulders slump]
Alright let's just go with your ((answer)) [he waves his hands then looks down at them briefly]
- 2.42 Shawn: What, twelve?
- 2.43 Dylan: Yeah, twelve.
- 2.44 Shawn: [writes 12 in blank]
- 2.45 Dylan: But I think.. naww. [sits down]
- 2.46 Shawn: I think I did it right. [writes 12 on his answer sheet and circles the second-highest confidence rating]
- 2.47 Dylan: I think it might be twenty-one.. I don't know.. I guess I'm just.. [writes 12 on his answer sheet, then places tip of his pencil on the second-highest confidence rating but quickly removes it and circles the middle rating instead]
- 2.48 Shawn: I'm not sure. [glances in the interviewer's direction and briefly chuckles]

Analysis and Discussion of Pair 2, Problem 2 — The social-cognitive breakdown occurred at lines 2.22 and 2.23, when Dylan proposed an incorrect strategy (adding all the numbers) while Shawn immediately launched into a correct one. Ideally, from a relational equity perspective, both students would have discussed their ideas and made an effort to understand the other person. Yet, here, Shawn pushed forward with his strategy and it was Dylan who engaged in the *social* as well as *cognitive* work of trying to make sense of Shawn's idea, with little help from Shawn (lines 2.25–2.36). We suggest that this moment highlights Shawn's relatively low commitment to Dylan's understanding of Shawn's strategy. Let us emphasize that we do not intend to pillory Shawn's behavior, because navigating (let alone *repairing*) these breakdowns requires potentially delicate, challenging forms of communication. It appears that both students were focused on task completion and therefore made their best effort to solve the problem as efficiently as possible. Stopping to “check for understanding” may not have been cued as a goal in this context. Dylan's posture and facial expression suggest that he had “given up,” and he gave the reins over completely to Shawn for the sake of finishing the task (line 2.41; see [Figure 2c](#)).

In Problem 2, as in Problem 1, both Shawn and Dylan reached correct solutions. However, this interaction was not equally beneficial for both students, at least in terms of their performance on the individual post-assessment: Shawn went on to solve all four equivalence problems and four out of six transfer problems correctly on the post-assessment, while Dylan solved only one equivalence problem and none of the transfer problems correctly. Shawn gained insight into the underlying structure of the problems and how to approach

them effectively. However, the quality of his interaction with Dylan, and Dylan's subsequent learning, left much to be desired from a relational equity perspective.

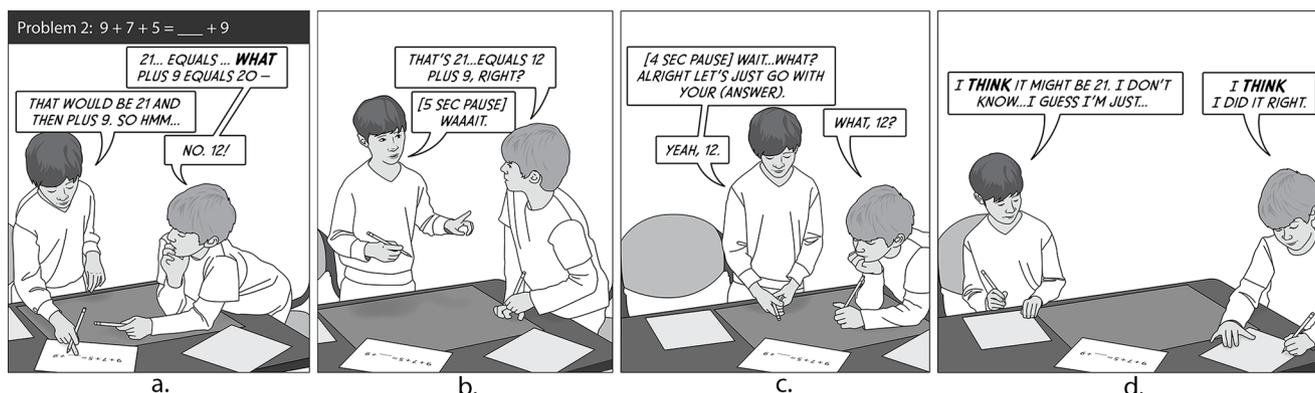


Figure 2. Synoptic Comic Strip of Dylan (Left) and Shawn (Right) Working on Problem 2. From Alibali, Gutiérrez, & Brown (2017), under a CC-BY4.0 license.

Pair 3: The Case of Marie and Jenny: Correct Strategies Adopted by Both Students

Jenny and Marie both used a correct strategy consistently at post-assessment, and both also increased their scores on the conceptual knowledge measure (from 0 to 1, out of 3, for both). We show in our analysis how this new mathematical understanding emerged as the result of their collaboration. Furthermore, we argue that the collaborative process through which this mathematical knowledge was co-constructed also manifested their respect and commitment to one another's learning.

Problem 1 ($8 + 5 + 4 = 4 + \underline{\quad}$) — In this case, both students noticed the location of the equal sign, which at first caused some confusion. Jenny and Marie spent multiple minutes trying to understand the problem format. Marie and Jenny started by working together to add the numbers on the left side of the equation (lines 3.1–3.5). When they reached the equal sign, both expressed confusion about how to proceed (e.g., line 3.7). After a while, Marie suggested an incorrect strategy (adding all the numbers in the problem, line 3.17), but Jenny redirected her attention to the equal sign and suggested that Marie's strategy did not honor the location of the equal sign (line 3.18; see Figure 3a). The two students promptly re-examined the problem. Marie then suddenly generated a correct strategy and successfully taught it to Jenny (see Figure 3b-d).

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| 3.1 | Jenny: | [glances at Marie then down at problem] Eight
//plus five is thirteen. |
| 3.2 | Marie: | //[looking at problem] Eight plus five..//
[nodding her head] Mmhmm. |
| 3.3 | Jenny: | And then it's four [slides her pencil to indicate the left "4"; shifts her gaze between problem and Marie]— |
| 3.4 | Marie: | Plus four—[as she adjusts in her seat, she quickly glances at Jenny then back to the problem] thirteen plus
//four is seventeen. |
| 3.5 | Jenny: | //Seventeen.//
[quickly shifts her gaze back and forth between Marie and the problem while tapping the middle of the problem with the tip of her pencil] |
| 3.6 | Marie: | [2 second pause; glances up at Jenny] And then.. [looks down at problem] seventeen... |
| 3.7 | J&M: | [20 second pause, during which Marie smiles, shrugs and raises her left hand, suggesting to Jenny "I don't know"; the two students look at each other and then back at the problem; Marie points with her pencil at the "4 =" |

- then looks back to Jenny, points again at the “4 + ___”; Jenny also then points with her pencil at the right 4; they shake their heads; Marie gestures toward the interviewer, perhaps suggesting they ask for help; Jenny shakes her head “no” to asking for help]
- 3.8 Jenny: So.. [gestures toward left $8 + 5$; in a low-voice] ((thirteen))—
- 3.9 Marie: [interrupting; gaze down] I don't get that part right there [indicates the equal sign]. So maybe it's like four [indicates left 4] equal to [indicates right side of the equation]—nooo [shakes head]. So that's seventeen [making an underlining gesture under the left side of the equation with her pencil], but then [indicates the equal sign]... equal to four.. [looks to Jenny]
- 3.10 Jenny: [looking at Marie, shakes head and whispers something inaudible]
- 3.11 Marie: Neither do I. [adjusts in her seat, glances at Jenny with a quizzical expression on her face, then leans in] Hmm..
- 3.12 Jenny: [looks down at the problem] ((What's the blank part?))..
- 3.13 Marie: So... [smiles, and looks between Jenny and the interviewer several times]
- 3.14 Jenny: [glances at the interviewer] ((Okay.)) [looks back at problem] That's thirteen [indicates the left $8 + 5$] and then that's seventeen [indicates the left 4]
- 3.15 Marie: Seventeen, yep, cause thir—three plus four equals seven [shakes her hands and head erratically as she speaks] so it's going to be seventeen (??).
- 3.16 Jenny: [looking at Marie] Okay. [shifts gaze back to problem]
- 3.17 Marie: And then, maybe we're supposed to add the four? [looks at Jenny as she indicates something on the right side]
- 3.18 Jenny: No it says [indicates the middle components of the equation] equals to four.
- 3.19 Marie: No it doesn't equal to four [looks at Jenny], because it's seventeen which it doesn't equal to four.
- 3.20 Jenny: [indicates the middle components of the equation again] Equals...
- 3.21 Marie: [looks back at problem] Four. No, ((well)).. Maybe it's like [leans back], so that's seventeen [indicates left 8; at this point, her speech speeds up considerably and she speaks with a tone that suggests she is excited] so four [indicates the “4” on the right side] plus [indicates plus sign] *what* [indicates the blank and briefly glances at Jenny] would make it seventeen [shifting her gaze back and forth between Jenny and problem; both hands open, palms facing up and slightly towards each other, gesturing to left side of gesture space] also [shifts her hands to right side of gesture space; looks at Jenny], which makes it equal together. [Fingers of each hand are together, and she brings them together, palms toward her chest, tips of fingers of the two hands touch]
- 3.22 Jenny: I don't get what you're saying.
- 3.23 Marie: So like.. that's seventeen together [gesture underlines left side of the equation with pencil] so four [points with her whole hand to the right 4] plus what [shifts her hand to gesture at the blank] would make that seventeen [fingers and thumb bunched together in a “wide pinch” position, gestured at the right side] so that these two [index points to the left 4 then the right 4] are equal together [gestures with her hands open, fingers spread, palms toward each other like she is holding something in front of her]. So since this is seventeen together [drags hand across entire left side of equation, her hand in a pinch shape, with her thumb underlining the equation and her other fingers tracing above the equation]
- 3.24 Jenny: Ohhh.. [releases the tension she was holding in her posture then leans back]
- 3.25 Marie: Four [indicates right 4] plus what [uses same gesture as before, on right side of equation] makes that equal together? Like, seventeen [indicates with pencil at the left side of the equation] plus four [indicates right 4] makes seventeen [indicates the blank],
//makes it equal [waves hands, both open and facing down, back and forth above the whole equation?]
- 3.26 Jenny: //So should we//
write thirteen? [Poises her pencil above the answer blank, looks at Marie]
- 3.27 Marie: Well I don't know if we're supposed to write it on there.
- 3.28 Jenny: So... [quickly moves to write on her answer sheet]
- 3.29 Marie: Let's write it in here [leans over to write on her answer sheet], that's what I think.
- 3.30 Jenny: [writes “13” on her answer sheet, looks back and forth between her answer sheet and the problem then circles the second-highest confidence rating]
- 3.31 Marie: Thirteeen. [writing “13” on her answer sheet] Okayyyy and.. [glances at Jenny's answer sheet] Umm.. [places pencil on the second-highest confidence rating then quickly moves it to the highest rating, but then quickly moves it back to the second-highest one and circles that]

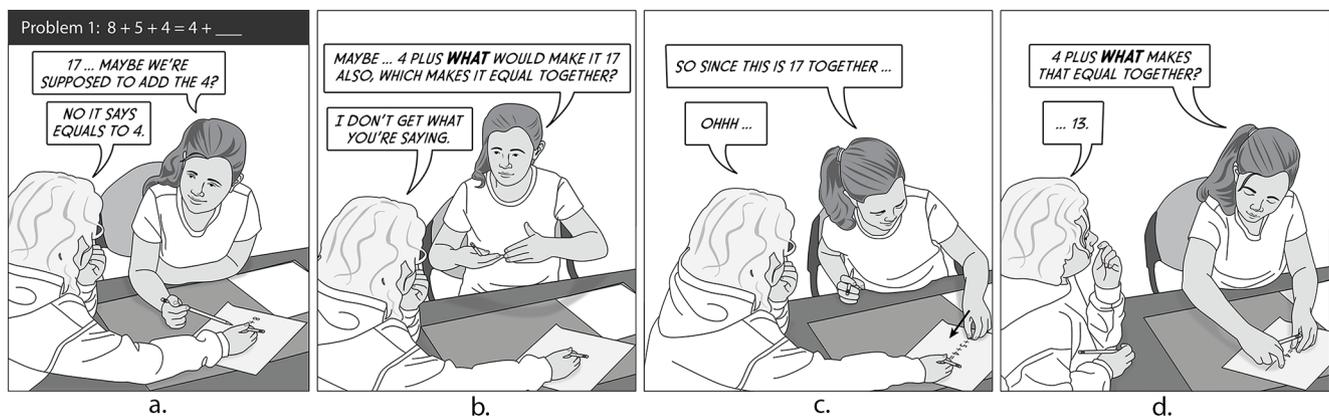


Figure 3. Synoptic Comic Strip of Jenny (Left) and Marie (Right) Working on Problem 1. From Alibali, Gutiérrez, & Brown (2017), under a CC-BY4.0 license.

Analysis and Discussion of Pair 3, Problem 1 — After initial confusion, Jenny and Marie worked together to find the final solution. Importantly, this final solution was derived from a *shared* mathematical understanding. They worked mostly in tandem, sometimes interrupting one another and in some cases finishing one another's sentences. For example, when looking at the left side of the equation, they both arrived at the answer of “seventeen” at the same time (lines 3.4 & 3.5). An important feature of their interaction, which contrasts with the other pairs, is that Jenny and Marie maintained their co-attention on one another as well as on the various elements in the problem space. Their gazes constantly shifted from each other, to the problem, and back to each other. In this way, both students played key roles in creating a space in which both of them could contribute and be heard, which led to the generation of the correct strategy.

When Marie suggested an incorrect “add-all” strategy (line 3.17: “Maybe we’re supposed to add the [right] four [to the left sum of 17]?”), Jenny immediately dissuaded her from this approach, and encouraged her to look at the problem, and specifically at the equal sign, again (line 3.18). This subtle shift in attention spurred Marie to re-examine the problem (line 3.19) and to discover a correct strategy. The manner in which Jenny responded to Marie’s proposed solution and the way Marie listened to this suggestion showed that each student respected the other’s contributions. Jenny did not initially understand Marie’s correct strategy, but when Jenny expressed her confusion (line 3.22), Marie offered an in-depth explanation with elaborate gestures, body movements, and even voice modulation (e.g., line 3.21: “four plus *what*”). The fact that Marie employed this vast array of resources suggests a strong commitment to Jenny’s learning, and indeed, Jenny understood the strategy after this explanation and even offered the answer before Marie did (line 3.26).

We view this pair as manifesting strong relational equity, in terms of both the respect and the commitment that they showed for each other. We regard *how* Marie successfully taught her friend, and not merely *that* she taught her, as noteworthy. We associate the elaborateness and effectiveness of Marie’s explanation with her commitment to Jenny’s learning. In this way, this pair differs from the case of Dylan and Shawn, for example, who were arguably more focused on task completion and who worked more independently of one another.

Problem 2 ($9 + 7 + 5 = \underline{\quad} + 9$) — The strategy that was articulated by Marie and Jenny during Problem 1 was robust enough that they were able to maintain it as a shared strategy and use it to solve Problem 2. In fact, they even generated a second correct strategy, the grouping strategy (see Table 1), which can be seen as a “short-cut,” and both Marie and Jenny understood the logic underlying this new strategy. The students collaboratively

solved Problem 2 very quickly, and they both correctly solved all four equivalence problems, as well as most of the transfer problems (Marie 4 out of 6, and Jenny 6 out of 6) at post-assessment. Thus, it seems clear that Marie and Jenny were both “on the same page” about these problems. This stands in contrast to Dylan and Shawn, who after Problem 1 still had markedly different understandings of mathematical equivalence problems, based on their collaboration on Problem 2 and their performance on the post-assessment.

In the transcript below, Marie and Jenny continue to act in concert and share the problem-solving space. They approached Problem 2 enthusiastically and expeditiously, both stating at the same time, “Like we did [it] the last time” (lines 3.39 & 3.40). Marie again used complex gestures and speech to articulate their shared, effective strategy (line 3.46).

3.32	Marie:	[looks down at problem; moves around in her seat as she talks] Alright. Nine plus..
3.33	Jenny:	[points to left 7 as she turns to look at Marie]
3.34	Marie:	[continues to randomly move in her seat as she looks at problem and talks] nine plus seven equaaals [looks at Jenny]—
3.35	Jenny:	[looking at Marie, finishing her sentence] Sixteen.
3.36	Marie:	[nods and looks at Jenny, then looks at problem, continues to squirm around in her seat] Sixteen. And sixteen plus five equaaaals twenty-one.
3.37	Jenny:	Yep. //And then—
3.38	Marie:	//And then.. Twenty-one ///[points to the blank using her pencil] plus—
3.39	Jenny:	///But like we// [vaguely gesture to blank] did it the last time. [looks at Marie]
3.40	Marie:	Like we did the last time. [glances quickly at Jenny as she’s talking] I bet it’s the same thing so—
3.41	Jenny:	[placing the tip of her pencil on right 9] Nine—
3.42	Marie:	[interjecting, quickly indicates the right + 9] Plus nine, so... uhh—What’s //seven plus [indicates left 7]—
3.43	Jenny:	//[looking at Marie] Uhhh// eleven. No!
3.44	Marie:	[looking at problem] Seven plus—
3.45	Jenny:	It’s twelve. [shifts her gaze between the problem and Marie] This will be twelve [pointing to the blank].
3.46	Marie:	[keeping her gaze on the problem] Well, let’s count it up. So seven plus five [covers up left 9 with her left thumb, and points to left 7 and left 5 with her right hand], what does that [cup shape under the left 7 and left 5, grouping the numbers together], cause the nine is already used [indicates left 9 then the right 9, then she covers both nines, one with each hand]. So [indicates left 7 and left 5] seven plus five equals?
3.47	Jenny:	Thirteen.
3.48	Marie:	Thirt—[pauses and looks at problem] No, seven //seven plus five ///equals twelve. [glances at Jenny]
3.49	Jenny:	///((That equals twelve.)) [gestures with her pencil vaguely to the left 7 and left 5]// ///Equals twelve.///
3.50	Marie:	So it’s twelve. [writes “12”, then circles second-highest confidence rating]
3.51	Jenny:	[writes “12” on her answer sheet; sees Marie circle the second-highest confidence rating and does the same; under her breath, to no one in particular, she says] Yeah.

Analysis and Discussion of Pair 3, Problem 2 — The respect and commitment to one another’s learning that Jenny and Marie manifested in their behaviors resulted in an efficient, authentic, and equitable interaction centered on a shared task. The students seemed to follow a single train of thought that proved fruitful. The co-attention that they established in Problem 1 carried over to Problem 2, where they again were able to finish each other’s statements (e.g., lines 3.37 & 3.38; see also their shared referents in lines 3.41–3.45). In fact, as Jenny and Marie started solving Problem 2, they both stated at the same time, “Like we did [it] the last time” (lines 3.39 & 3.40). Their stated agreement on what strategy to use indicates that they had shared ownership of the mathematics, which reflects the respect and commitment they had for each other. Marie and Jenny actually co-generated a new strategy (the Grouping strategy, see Table 1). Previously in Problem 1, they made both sides sum to the same total (the Equalize strategy, see Table 1), but they solved Problem 2 by noting that two of the addends are already equal (in this case, the two 9s; see line 3.46), so one need only add the other addends (in this case, 7 and 5) to arrive at a solution.

Both Marie and Jenny drew on their new, shared knowledge of the task to solve nearly all of the equivalence and transfer problems correctly at post-assessment, demonstrating that their newly constructed knowledge was quite robust. In this way, their actions both manifested strong relational equity *and* promoted their generation and adoption of two correct strategies.

Discussion

In this research, we combined theory and methods from psychology and mathematics education to gain new insights into processes of strategy change and learning. We adopted the lens of relational equity and used it to theorize about the ways in which individual mathematics learning and social relations that manifest respect and commitment are mutually constituted in, and achieved simultaneously through, interaction. We applied this theoretical perspective in a specific mathematical context—mathematical equivalence—that is undergirded by more than a decade’s worth of findings from developmental and educational psychology.

This conceptualization—that mathematical knowledge and relational equity are co-constructed— applies to the ways the students in our study responded to one another’s mathematical perceptions and actions. Morgan’s and Dylan’s contributions, for example, were not always taken up and discussed with their partners. In this sense, these students’ personal interpretations of the equal sign were “forced out” of the conversation. The limited respect and commitment in the first two cases is striking when compared to the third case. In this pair, Marie sensitively responded to the fact that Jenny was falling behind. Moreover, Marie began to see the equal sign as a symbol expressing a relation between quantities before Jenny did, setting up the possibility that she might generate a new approach to solving the problems, and Jenny would not. Yet Marie demonstrated a strong commitment to Jenny’s learning, and she used an elaborate array of communicative means to explain the new strategy that she generated. This strong commitment and respect was not found with Dylan and Shawn, or with Morgan and Elsa. Indeed, issues of relational equity appeared to play out in a different way in each pair. These data illuminate the challenge and the opportunity inherent in navigating these micro-processes.

We view *all* instances of peer collaboration as manifesting some form of relational equity. With this guiding conception, we place no value judgment on the students’ behaviors, in particular because the empirical context

we studied was not set up with the aim of promoting relational equity (Boaler, 2008). Instead, we seek to understand the resources, intuitive or explicit, that students bring to peer collaborations. The present findings suggest that this effort will yield deeper understanding of processes of strategy change and learning, and at the same time will enable us to identify potential leverage points for supporting the development of respect and commitment through targeted pedagogical interventions.

Limitations

Of course, there are many limitations to the present investigation. Notably, our sample—both the full sample of 38 pairs and the focal sample of three pairs—was relatively homogeneous in terms of race, class, and ethnicity. These dimensions of social identity are undoubtedly influential in shaping patterns of relational equity in interaction. Yet with the current dataset, we are unable to address these issues directly. However, we believe that the connections between social identity and relational equity are an important arena for future work.

Second, all of the pairs in our study were made up of friends, but we have no data on the length or quality of these friendships. We are therefore unable to investigate how aspects of their relationships affected the emergence of relational equity. A related limitation is that the pairs were constructed by the participants themselves. The members of each pair knew one another and were comfortable with each other, but the pairs were not constructed with an eye towards promoting mathematics learning or relational equity (Boaler, 2008; Boaler & Staples, 2008). A purposeful focus on pairs of strangers or pairs with differing levels of initial knowledge might yield different results.

Of course, in many instructional situations, students have a choice about whom to work with, and in such cases, students typically choose to work with a friend. Understanding the dynamics of friends working together is important for informing instructional decisions about whether to assign “math partners” or to allow students free choice. In this sense, our focus on friends working together represents a strength as well as a limitation. This work is a first step at characterizing the micro-processes involved in friends working together. Future studies could compare patterns of relational equity and learning in students working with friends and with unfamiliar peers.

A third limitation is that our approach does not support generalization in the statistical sense (i.e., making claims about a population based on a sample), which is often a goal of psychological investigations. The selection of the three focal cases from our larger sample was not random, nor was our sample of participants representative of the population of elementary students as a whole. In this regard, it is important to note that we do not seek to generalize our results from the three focal cases to all elementary students, or even to all 76 participants in the present sample. Instead, our qualitative micro-analysis of the three pairs was intended to shed light on how mathematical ideas and relational equity are co-constructed during peer collaboration, and on how both social and cognitive factors contribute to students’ generating and taking up new ideas. Additional work that combines psychological and mathematics education perspectives is needed to understand the range of circumstances in which these processes occur, to guide generalization beyond the scope of the current study.

Finally, our laboratory-based experimental approach also constrains our interpretations in many ways. We could draw evidence regarding respect and commitment only from the admittedly small envelope of the collaboration session. The laboratory setting was not set up to promote relational equity *a priori*, and we do not

claim to be able to generalize to other settings. We believe, for instance, that Shawn and Dylan could have a highly respectful interaction in another space.

The laboratory context also differs in many ways from peer collaborations as they might occur in authentic classroom contexts. This had the potential benefit of preventing off-task behavior among our participants; however, this unnaturalness may also have constrained students' interactions in other, less positive ways. Given the unnatural context, we are not surprised by the limited relational equity manifested in Shawn and Dylan's collaboration. On the contrary, we suggest that, within the constraints imposed by the laboratory setting, the strong relational equity displayed by Jenny and Marie is quite remarkable.

Contributions and Conclusions

From the perspective of psychology, our work contributes to a deeper understanding of processes of strategy change. Our micro-analytic approach reveals processes that are rarely the focus of laboratory studies, which tend to focus on broader trends in larger samples of students. Indeed, had we created profiles of students based solely on their use of "correct" versus "incorrect" mathematical strategies, our three cases would have fallen into different categories and the details of their learning (or lack thereof) would have remained buried in the aggregate data. We suggest that the interactional dynamics influenced whether and how students discovered and took up correct strategies, even after they noticed the location of the equal sign. As our data show, it is not necessarily true that noticing the equal sign leads students unequivocally and automatically to generating or adopting correct strategies. Instead, interactional dynamics influence whether students actually use the available "raw material" (i.e., their noticing of the location of the equal sign) to construct new, correct ways to solve the problems.

From the view of mathematics education, our work contributes to understanding how the social and cognitive dimensions of interactions around mathematics are both inseparable and mutually supportive. Respect for others and commitment to others' learning are indeed laudable objectives for pedagogy. Our research underscores the need for greater understanding of how these values manifest in interactions and how we can support them. A better understanding of respect and commitment in interactions about mathematics also has implications for understanding trajectories of identities and engagement in mathematics over the longer term (see, e.g., [Langer-Osuna, 2011](#); [Wood, 2013](#)). Thus, investigations into the reciprocal relationship between relational equity and mathematics learning can target both opportunities to learn rigorous mathematical content, and issues of identity and power—in R. Gutiérrez's ([2009, 2012](#)) terms, both dominant and critical axes of mathematics education.

The real power of our findings comes from combining psychological and mathematics education perspectives. To deeply understand processes of strategy change in mathematics learning, we must understand how specific "moves" in social interactions manifest *both* the cognitive elements that are in play at a given moment (e.g., noticing of problem features, proposals for solution strategies) *and* the social dimensions that influence whether those elements will be taken up or disregarded. Our work suggests that the social dimensions of respect and commitment are mutually co-constructed, simultaneously, with mathematical knowledge, through the micro-processes involved in one-on-one interactions.

Notes

i) According to Boaler (2008), teachers using Complex Instruction modeled complex ways of communicating about mathematics that students, in turn, learned to use themselves. Forms of communication such as asking probing questions, rather than simply “telling” the answer, and drawing connections about mathematics more broadly are emphasized as prominent features of the equitable teaching approach. As we elaborate in the Methods section, our laboratory setting did not involve pedagogical interventions aimed at teaching these communication skills. Instead, our analysis is focused on acts of communication that occurred, in their entirety, during interactions in this laboratory setting.

ii) According to Boaler (2008), teachers using Complex Instruction took great effort not only to teach students to take responsibility for each other’s learning, but also to *value* that responsibility “as an important part of life” (p. 178). We wish to acknowledge this important aspect of relational equity and to clarify, once again, that our laboratory protocols did not involve such pedagogical interventions. That said, the collaboration session did include an element of shared responsibility at the level of interaction, given that the directions given at the outset of the collaboration episode encouraged students to work together.

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Competing Interests

The authors have declared that no competing interests exist.

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Appendix

Table A.1

Focal Data (Six Participants) With Pre-Assessment, Post-Assessment, and Transfer Scores.

Pair	Pre-Assessment Equivalence Problems Score (out of 4)	Post-Assessment Equivalence Problems Score (out of 4)	Post-Assessment Transfer Score (out of 6)
Elsa	0	0	0
Morgan	0	0	1
Dylan	0	1	0
Shawn	0	4	4
Jenny	0	4	6
Marie	0	3	4