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Re-Thinking 'Normal' Development in the Early Learning of Number

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Abstract

In this article we suggest that, notwithstanding noted differences, one unmarked similarity across psychology and mathematics education is the continued dominance of the view that there is a 'normal' path of development. We focus particularly on the case of the early learning of number and point to evidence that puts into question the dominant narrative of how number sense develops through the concrete and the cardinal. Recent neuroscience findings have raised the potential significance of ordinal approaches to learning number, which in privileging the symbolic—and hence the abstract—reverse one aspect of the 'normal' development order. We draw on empirical evidence to suggest that what children can do, and in what order, is sensitive to, among other things, the curriculum approach—and also the tools they have at their disposition. We draw out implications from our work for curriculum organisation in the early years of schooling, to disrupt taken-for-granted paths.

Keywords: early number, development, mathematics, teaching, technology

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The Developmental Narrative in Psychology and Learning Number

Following Piaget, it is perhaps an easy temptation to interpret sequences of child behaviour in terms of a developmental path both in general and, in our particular concern in this article, when considering the learning of number. Despite differences between the fields of psychology and mathematics education, we see one similarity (which perhaps goes unmarked) in the use of development or trajectory metaphors to explain behaviour. A good example of such logic is from the first volume of this journal: 'The results of the current study revealed a clear developmental pattern through which preschoolers traverse towards Arabic digit knowledge' (Knudsen et al., 2015, p. 21), a conclusion reached as a result of interpreting children's responses to a series of tasks. We can understand the interest that both cognitive scientists and mathematics educators might have in identifying developmental progressions, however, we would like to call into question the very notion of a 'natural' or 'normal' developmental sequence. We question, for example, whether tasks framed in a different manner might have 'revealed' different patterns of 'development' (Hughes & Donaldson, 1979). We also want to ask what influence the concept itself (which is being learnt) might have on what children can do and in what order?

In the context of early childhood education, Fowler (2017) argues that tensions around the use of developmental theory arise in part from the different types of development in question. In the Piagetian approach, in which development drives learning, the focus is on development of “universals” such as object permanence and speech. In the Vygotskian approach, where learning drives development, the focus is on learning non-universals, such as reading and counting. In recommending that teachers adopt a multi-dimensional framework, in which they are aware of how the relationship varies between learning and development within the universal versus non-universal developmental sequence, Fowler does not disrupt the notion of a normal developmental path. This, despite the fact that Vygotsky’s approach more radically insists on the role of tools—of language, symbols, technologies—in driving development, tools that can be significantly different in the context of early number learning, for example, especially with the advent of digital technologies. From such a point of view, the tools (and accompanying tasks) are the pivot on which learning happens, and the source of major potential diversity in development, thereby challenging the notion that there is a ‘natural’ development (unless that development occurs in the context of using ‘natural’ tools!).

The potential for diversity in development has also been advanced, but for different reasons, within psychology, where there are critiques of the view of normative development (e.g., O’Dell & Brownlow, 2015) particularly when concerning ‘neurodiverse’ students (e.g., Fenton & Krahn, 2009; Trott, 2015). From a critical pedagogy perspective (Kincheloe & McLaren, 1994) that brings social and ideological dimensions into consideration, ‘normal’ development is a convenient social construction that operates by means of distinction from ‘others’ who are taken as different from the norm. Rose (2003) suggested that we are entering a new era in which the self is seen in neurochemical terms. Undesirable behaviours (such as alcoholism) need no longer be seen as signs of moral corruption but more simply as an indicator of a neuronal imbalance that can potentially be altered by a drug. He suggests ‘we may be seeing the emergence of a new way of thinking: *variation without a norm* and perhaps, even, *anomaly with abnormality’* (p. 22). We want to suggest that it is worthwhile examining the assumption that there exists a normal developmental pathway from which others who ‘deviate’ are considered abnormal. This would require a different way of thinking about development, but also a different understanding of mathematical concepts, which could accommodate more deviation that is currently assumed. To explore these ideas, we firstly consider the evidence for the existence of typical developmental pathways in mathematics in general and then specifically for the case of number.

‘Normal’ Development in Learning Mathematics

There are at least two different senses in which ‘development’ or ‘trajectories’ are used in relation to mathematics. Firstly, it is possible to categorise students into groups who display similar development or growth patterns of overall mathematical attainment (Salaschek, Zeuch, & Souvignier, 2014, p. 104). For example, one strong pattern, internationally, is that students who underachieve in mathematics at an early age tend to continue under-achieving (Bradbury et al., 2015).

The second use of ‘trajectory’ is in relation to particular mathematical skills. Work done in the 1980s (Denvir & Brown, 1986) suggested there was a descriptive framework that could aid the assessment and remediation of children’s competence with number, by accurately capturing the order in which children acquired skills. Further examples come from the work of Fuson (1992) who elaborated different ‘levels’ which children move through in learning number. There have been many articulations of similarly numbered levels in relation to counting (e.g., Ashcraft, 1982; Baroody, 1985; Baroody & Wilkins, 1999; Carpenter & Moser, 1982, 1984) and multiplication (e.g., Anghileri, 1989). The direct implication of such frameworks is that children must start learning about number and addition through manipulation of concrete objects and that, for example, treatment of multiplication and division must wait until sufficient progress has been made in counting, addition and subtraction.

Simon (1995, p. 136) introduced the term ‘hypothetical learning trajectory’ to capture the idea of a teacher’s prediction of students’ ‘expected tendency’ (p. 136). There is an assumption of some ‘regularity’ (p. 136) to individual’s learning, whilst recognizing the idiosyncrasy of each particular journey. Work on learning trajectories has been taken up by other authors, particularly in the USA, linked to the Common Core Standards (Confrey, Maloney, & Corley, 2014). In this work, trajectories ‘document in detail the likely progressions, over long periods of time, of students’ reasoning about big ideas in mathematics’ (p. 720). Authors in this field are often keen to distance themselves from the kinds of developmental perspective reviewed above, for example:

> Learning trajectories are not a stage approach (Piaget …), which delineates developmental stages that must be mastered before passage to later stages. Rather, they are probabilistic statements that claim that, given rich tasks and tools carefully sequenced to build from prior knowledge, students tend to exhibit predictable ranges of behaviors, including their responses to the tasks and their ways of speaking about or explaining their reasoning. (Confrey, Maloney, & Corley, 2014, p. 721)

However, despite the claims that such work is not about delineating developmental stages, it is clear even in the quotation above that there *is* an assumption about a normal pattern of learning and development, given a particular context of tasks and tools. We suggest there is a circularity here that comes from switching between two uses of the word development. Learning trajectories are partly based on research evidence of typical development (Confrey, Maloney, & Corley, 2014), for example, from quantitative analysis of large samples of students (e.g., Confrey et al., 2009) but they are then used to design teaching sequences, moving to a more particular meaning of development for an individual. If hypothetical trajectories are used to design teaching sequences it is perhaps likely students *will* follow predicted pathways. However, this is a little bit like the scenario of a model being used to design a research instrument whose results are used as evidence for the applicability of the model. What is potentially missed, therefore, is the possibility of approaches other than those built into the original research upon with the learning trajectories are based, or outside the experience of the researchers who also add their logical analysis of the curriculum, in devising trajectories. For example, in a description of a trajectory of counting (Daro, Mosher, & Corcoran, 2011, p. 67) an exclusively cardinal description of counting is offered (i.e., counting is only seen in relation to counting objects, ignoring the more ordinal, intransitive counting). The specific trajectories themselves end up looking a lot like the earlier work of developmental stages, levels and progressions, even though the authors claim not to be caught in the same assumptions.

The specifics of the learning trajectory for counting leads us to consider in more detail the case of early number learning. There have been alternative curricula articulated (Davydov, 1990; Gattegno, 1974) which disrupt the kinds of ‘typical’ developmental pattern articulated by Fuson (1992) that continue to be represented in learning trajectories, around number. In these curricula, children work with ideas of algebra before arithmetic (Gattegno, 1974), or proportion before addition (Davydov, 1990), or of division (in terms of ‘splitting’) and multiplication before addition and subtraction (Confrey, 1994). Note that in each case, the concept in question does not remain static, coupled as it is with particular tasks and tools that enable children's learning. The ideas of Davydov in particular have received recent attention (Coles, 2017; Dougherty, 2008; Savard, 2017), yet have not gained widespread acceptance and have not challenged the narrative of the inevitability of children progressing through predictable hierarchies of achievement.

One particular feature of the dominant view of early number, alluded to above, is that learning must begin with the cardinal (the aspect of number linked to describing the numerosity of a set of objects), as can be seen in Cross, Woods, and Schweingruber (2009). Not so long ago, however, debate over the primacy of cardinality over a more ordinal approach was lively amongst mathematicians, philosophers of mathematics and psychologists. Ordinality refers to an aspect of number or numbering (first, second, third, …) that connects to the temporal in the sense that it is about order (what comes next?) rather than size (how many?). Ordinal relations seem to be between numbers themselves (what it means to be 4^{th}, is that it is after 3 and before 5) rather than in relation to objects. In mathematics, both Peano and Dedekind argued for the primacy of ordinals, whereas Russell advocated for cardinals. Brainerd (1979) asserts that Piaget ignored the logical distinctions underlying these two number variations and used his experimental evidence as a basis for combining the ordinal and cardinal. After Piaget, who may well have espoused a balanced approach to number, the work of Gelman and colleagues (e.g., Gelman & Meck, 1983) elaborated an almost exclusively cardinal conception of number, which has gained widespread acceptance in the field, continuing to this day (and in evidence in the frameworks referenced above). It is not clear to us why ordinality plays such a relatively small role in current thinking about early number learning, and we wonder if it is related to Piaget's theories around abstraction, linked to the assumption that counting 'how many' things there are (and therefore focusing on cardinality) is more 'concrete' than working with symbols and their relations to each other (the focus of ordinality) (see Tahta, 1991, 1998). There may also be the influence of set theory, which is a branch of mathematical logic that deals with collections of objects and in which concepts such as one-to-one correspondence are highly useful. Set theory was dominant at the time in the discipline of mathematics, and propelled the 'new math' movement.

In the next section, we set out to question the assumption that a concrete-cardinal basis for number leads to more symbolic, ordinal, abstract and relational uses. We argue that, at the least, such a pathway of learning is highly context dependent. By focusing on ordinality-cardinality, we aim to offer a particular critique that also speaks more generally to the 'typical' developmental assumption in learning. We draw initially on recent neuroscience in the next section and then some empirical data from our own work in primary schools in Canada and the UK, which involves the use of markedly different tools, and associated tasks, than those typically used in learning trajectory research.

Questioning Assumptions Around Ordinality-Cardinality in the Early Acquisition of Number Sense

For neuroscience researchers, *number sense* is a concept that has been operationalized through the identification of neurological changes that can be correlated with brain activities that occur when people work on particular numerical tasks. However, a particular issue raised in a review of work connecting neuroscience and mathematics education research (Ansari & Lyons, 2016) is the continuing problem of how to translate differences in neural patterns, firstly into behavioural terms, and then into educationally relevant concepts (p.

381). This problem is exacerbated by the fact that (presumably largely for ethical reasons) many neuroscience studies are carried out on adults (p. 380) raising questions of the ecological validity of relating findings to school age children.

Prevalence of Cardinality in Neuroscience Research

Through integrating results of a range of studies, in this section we argue that the concepts of 'ordinality' and 'cardinality' are ones that fulfil the criteria from Ansari and Lyons (2016) of being educationally relevant and related to both behavioural and neural effects. We begin with an influential background study. Dehaene et al. (2003), based on research in neuroscience, introduced the idea of a triple-code model of number, a model that continues to inform current work (e.g., Sokolowski et al., 2017). In the triple-code model, number is seen to comprise:

1. A visual Arabic code in which numbers are represented as sequence of digits.
2. An analogical quantity or magnitude code.
3. A verbal code in which numbers are represented as a sequence of words.

They suggest that understanding of number involves two major process of transcoding, from (1) directly to (3) (which they call asemantic) and a semantic route from (1) to (2) to (3). The label 'semantic' (indicating 'meaning', presumably) is reserved only for the sense of number that incorporates magnitude, code (2). The assumption here seems to be that meaning (semantics) comes from magnitude and we see in the work of Dehaene et al. an emphasis on quantity (and cardinality) as the basis for the meaning of number that is common in the field of educational neuroscience.

The majority of work that takes place in neuroscience and early number learning emphasises this cardinal, quantity-oriented aspect of counting and numeral reference (Vogel et al., 2015). There is widespread recognition that humans share aspects of brain structure with other animals that are central to subitising and working with small numerosities, our Approximate Number System (ANS) (e.g., Butterworth, 2005). It was generally assumed that there is a gradual transition from use of the ANS to the achievement of a more fully formed awareness of number and place value (Nieder & Dehaene, 2009). A recent meta-study suggested evidence of a relationship between the ANS and symbolic number skill; however, we are still far from understanding any underlying mechanisms that drive this relationship (Szkudlarek & Brannon, 2017).

Challenges to the Dominance of Cardinality

In the last decade, increasing numbers of researchers have challenged the dominant cardinal view of number cognition, and have investigated tasks that engage ordinal or relational thinking (e.g., Schalk et al., 2016). For example, Lyons and Beilock (2011) created an ordinal task that consists of a sequence of three numerals (instead of symbols, dots can be used as well). Participants are asked to decide whether the numerals are in the correct order (they can be ascending or descending). For example, both the sequences [3, 4, 5] and [5, 4, 3] would be considered to be in the correct order, but not the sequence [3, 5, 4]. In a behavioural study (in Grades 1 to 6), speed and success on ordering tasks like the one described above was strongly correlated to wider mathematical achievement (Lyons et al., 2014) once children were in Grade 2 or above. These experiments suggest that for students who are successful at mathematics a significant aspect of the meaning of a numeral is relational and strongly tied to the unfolding of the *sequence* of numerals. There was a much smaller correlation with overall achievement in the case of ordinal interpretations of dots. For this task, participants again had to judge whether three groups of dots were correctly ordered or not (e.g., from left to right, 2 dots, 4 dots, 5 dots are in order; 2 dots, 5 dots, 3 dots are not).

The well-established 'distance effect' (for tasks with cardinal numbers, people are quicker to identify the larger of two numbers when they are further apart) can be seen in another behavioural distinction between tasks that involve dots and those that involve numerals relates. As Lyons and Beilock (2011) show, the distance effect can also be seen in judgements of order when the terms are given as dots but, when the terms are given as numerals, there is a reversal of the distance effect. In other words, participants are quicker at deciding whether or not three numerals are in the correct order the closer they are together. Based on this finding, Lyons and Beilock suggest that the brain is engaged in a different kind of activity when comparing the ordinality of numerals; different than when comparing cardinality (of numerals or dots) and when comparing ordinality using dots. However, it is important to note this study was conducted on adult participants with an average age of twenty.

In trying to make sense of this 'reversed distance effect' in ordinal compared to cardinal processing, it is plausible that ordinal tasks involving numerals make use of the memorised number names of the "number song". This is a hypothesis that is consistent with Seidenberg's (1962) theory of the ritual origins of counting. According to Seidenberg, the ordered naming of numbers precedes, historically speaking, the use of numbers as tools to determine quantities (that is, to figure out how many cows or siblings one has). From this point of view, ordinal counting involves calling forth names (which were originally the names of Gods, which then become the actual names of numbers) one after another. (For more on this, see Sinclair and Pimm, 2015).

In research focused on developmental dyscalculia (DD), Rubinsten and Sury (2011) use the same task as Lyons and Beilock (both symbols and dots) with typically developing adults as well as adults with DD and found that both groups performed in similar ways on the symbolic tasks, but differed on the dot task[i]. They suggest that linguistic knowledge may facilitate ordinal number processing, which is consistent with Seidenberg's hypothesis and also revelatory of the difficulty of isolating any component of number sense, be it ordinal or cardinal. The significance of the number song and language point to a powerful temporal sense of number. Schalk et al. (2016) report on a study that followed children from Grades 1 to 3 and found a correlation between the capacity for relational reasoning (assessed via continuing patterns in numbers of objects) and mathematical achievement tested two and a half years later. This study again provides behavioural evidence for the idea that relational judgments *between* quantities (which are linked to ordinality) are significant for mathematical success.

Moving onto links between ordinality-cardinality and neural effects, Lyons and Beilock (2013) have used functional magnetic resonance imaging (fMRI) scans to continue their exploration of differences in number processing. In particular, they found that there seem to be connections between: (i) cardinal processing of dots; (ii) cardinal processing of numerals; (iii) ordinal processing of dots; but that, in contrast, the ordinal processing of symbols is different and the 'odd one out' – as it was in their behavioural studies. In their summary, they state:

> Overall, these data are consistent with the notion that assessing ordinality in symbolic and nonsymbolic numbers relies on qualitatively different processes. Overlap was observed only in a prefrontal area not canonically associated with basic number representation. (p. 17057)

It is important to note that fMRI images highlighting regions of the brain are only illustrating *differences* in brain activity during a task, compared with a control task. In other words, there are large regions of the brain also active, but what is flagged up are those regions that change activation pattern in a statistically significantly manner (by detecting changes associated with blood flow to the brain) during the different tasks. It therefore does not seem to us entirely accurate to locate number processing in a particular brain region, or at least it must remain a possibility that there are vital elements of number processing taking place in distributed parts of the brain (which are also active when not doing number work). Nonetheless, it still seems reasonable to conclude that there are some differences in brain processing when engaging in the ordinal comparison of numerals compared with the other types of number processing tested. These results, although carried out on university age students, are suggestive. At present, no studies have tested a similar result with younger children (Vogel et al., 2015, p. 35), although Vogel et al. (2015) have behavioural evidence, from grade one children, to suggest a difference in processing of symbol-symbol number judgments compared to symbol-quantity judgments.

A hypothesis emerging from the studies reviewed above is that: (a) ordinal processing of numerals is distinct from other aspects of number sense; and, (b) ordinal processing of numerals is correlated with broader mathematical success. If there is validity in these conclusions, then a direct implication is that a key in learning number sense is becoming aware of how number symbols relate to each other (rather than their links to objects). In other words, although still limited by the tools (e.g., fMRI) used to identify brain response, which can only capture static images of brain activity, we see the new findings by Lyons and colleagues as drawing attention to relational and symbolic aspects of number. We are mindful of the dangers of making the kinds of reductive assumptions that are sometimes required of neuroscientific research. Nevertheless, the attention that is being brought to ordinality in this research could help support research both in psychology and in mathematics education because it highlights the significant role that task design plays in the process of defining what will count as number sense. It also draws attention to under-valued aspects of number sense in the primary school curriculum.

Summary of Results

Putting the conclusions of the research reviewed in this section together, we suggest that ordinality and cardinality are educationally relevant concepts (Ansari & Lyons, 2016) that a range of studies have shown are closely linked to behavioural effects and, with adults, neural effects. The studies we reviewed point to evidence that a mature number sense is signalled by symbolic ordinal awareness. The question we want to raise, therefore, is whether focusing almost exclusively on cardinal and magnitude aspects of number in the early years of schooling is the best way of achieving the kind of number awareness that seems necessary for success in mathematics? Might it be the case that the developmental trajectories noticed by researchers are a result of assumptions built into curricula and assessment tasks that do not allow any other possibility for children’s learning? The fact that cardinal and ordinal skill with number appear to call on different brain patterns complicates the picture in terms of whether there is a ‘natural’ path of development. In the next section, we move to empirical data which suggests that, indeed, when presented with different types of curriculum ordering, children show evidence of learning about number in surprising ways.

Empirical Questionings

In this section, we draw on data from joint research we have been doing in the UK and Canada, in which we have collaborated on designing classroom tasks and the analysis of data. Part of this work has been a questioning of what it means to understand place value. By way of context, we briefly sketch our thinking around place value—and how it relates to the issues around number sense we have been discussing in the previous sections—and then offer our methodology and some illustrative data and analyse the children's actions in terms of evidence of number sense.

In the early years of schooling in the UK and Canada and in teacher education in these countries, the focus of work is on the magnitude code. As one teacher in the UK we have worked with put it, the orthodoxy is that young children must work with numbers that are 'graspable', which means working with beads, counters and one-to-one correspondence. In terms of place value, this approach evolves into working with similarly 'graspable' metaphors for number such as base ten Dienes blocks. In the USA, the Principles and Standards for School Mathematics recommends that children in grades pre-K–2 should "use multiple models to develop initial understandings of place value and the base-ten system" (NCTM, 2000, p. 78) and the Common Core State Standards recommend "using concrete models or drawings" to add and subtract within 1000. Similarly, Ross (2002) argues for the use of digit-correspondence tasks in which "students are asked to construct meaning for the individual digits in a multi-digit numeral by matching the digits to quantities in a collection of objects" (p. 420).

Recently, Pimm (2018) has questioned where place value exists across all three of Dehaene et al.'s codes (see Figure 1).

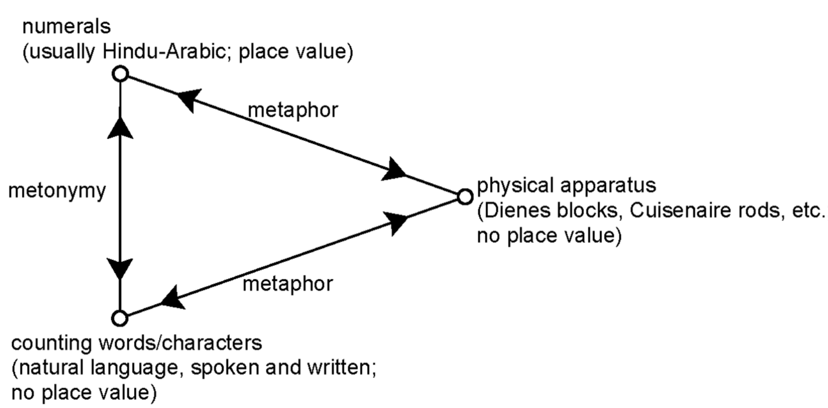

Figure 1. Reproduced from Pimm (2018).

Pimm suggests that place value (in the sense that it is the 'place' of a numeral in a sequence that determines its value) only exists in written numeral form (he also points to the different kinds of relations (metaphor or metonymy) that exist between the three codes, which is not our focus here – for more on that distinction in this context, see Tahta, 1991). If we are dealing with blocks or beads or apparatus, *where* objects are 'placed' is irrelevant to their value. In spoken language, it is not the place in a string of words that gives value (and e.g., in German, '24' is spoken as 'four and twenty'). It is only in the written numeral form (e.g., '24') that the 'place' holds the value. We have found Pimm's arguments to be so counter to entrenched ways of thinking that they are hard to comprehend at first. We are perhaps so used to assuming that number simply ‘has’ place value that it is hard even to question. What we see Pimm as arguing is, for example, that if you throw two ‘ten’ blocks and four ‘unit’ blocks on the floor then, wherever they land, you will have twenty-four units. The grouping (into tens) makes the totals easier to count but not due to place value, but because of our base-10 system (e.g., representing the total in base 9, the usefulness of the grouping into 10s falls away).

We want to ask, how else is it possible to understand the place value of written numerals other than through links to objects? We do not want to argue that the quantity code is unimportant: however, we do suggest that there is much to be gained from exploring an increased focus on the verbal code (and, with it, the temporal) and its links to written systems. How might different tools change the concept of place value, as well as the opportunities for students to learn about place value? In our current research, we have been exploring this question using two different kinds of tool, each of which we argue offer alternate pathways to the learning of place value in primary school and that disrupt the predominant development of the concept that begins with the concrete. One tool is an iPad app *TouchCounts* and the other the ‘Gattegno Tens Chart’, both of which are explained below. Our research interest is in studying how children learn in novel situations. We focus less on comparisons with traditional pencil-and-paper activities or even manipulatives, especially since these tend to be cardinal in nature—which would make comparison invalid. As we explain below, our methodological perspective prompts attention to the children’s tool-specific, emergent conceptualisations.

Methodological Considerations

Inclusive materialism, as developed by de Freitas and Sinclair (2014), provides a way of attending to the sociocultural conditions of learning, while at the same time allowing a role for the body and the physical environment in mathematics teaching and learning. It does so by adopting a monist position that accords the same ontological status to mathematical concepts, human bodies, discourses and physical objects. A first consequence is that mathematical concepts are no longer seen as abstractions of sensori-motor experiences, but inevitably entangled with those experiences. In other words, a number is not a Platonic ideal, nor is it merely a sociocultural creation. Rather, it is an assemblage of counting fingers, things-to-be-counted, words to count with, and so on. In this view, the idea of the human body as being well-defined by the contours of the skin melt away as the body is both expanded to include physical tools and objects, as well as contracted, perhaps even to a single neuron. We believe that this approach enables us to draw on neuroscience research in profitable ways, without discounting socio-political dynamics. A methodological consequence of this theoretical orientation is that concepts are seen as being entangled with devices (be they fingers, manipulatives or digital tools), so that devices such as Dienes blocks are not simple mediators of meaning that can then be discarded, or divorced from the concept. From this point of view, any talk of a learning progression *must* include the device, as well as the associated tasks.

Another methodological influence on us is an approach that has roots in biology (Maturana & Varela, 1987) and neuroscience (Varela, 1996) and similarly attempts to bridge first- and third-person perspectives—enactivism (e.g., see Reid & Mgombelo, 2015). From the enactive perspective, the architecture of the brain is seen as one aspect of the multitude of relations between the components that constitute our being (that includes any tools we might use). This web of relations is labelled our ‘structure’. Every interaction in the world alters our structure and one of the enactive insights is that humans are ‘structure-determined’ beings. In other words, when an event occurs which provokes a response, the response we give is not a function of the trigger but a function of our structure.

Bateson (1972, p. 409) famously gave the example of kicking a dog – where what happens next is not determined by your kick, but by the structure of the dog itself, which includes the history of your interactions with that dog. The cultural and social are embodied in our very beings, in our structure. As a result of the history of our interactions, in most situations we make automatic responses – from driving a car to the small prejudices we may catch ourselves projecting onto others who are not like us. As a result of commonalities across human DNA it is not surprising that some aspects of our structures may have similar features – for example similar areas of the brain becoming active during number tasks. However, enactivism is committed to the view that what is significant in our structure is the set of *relations* between components, not the components themselves (Maturana & Varela, 1987).

Common across both inclusive materialism and enactivism, therefore, is the view that an individual does not end at the boundary of the skin and that the social, cultural and political are enmeshed in the physicality of each of us (Bateson, 1972; de Freitas & Sinclair, 2014). From the perspective of these stances, if considering the learning of mathematics by children in school, we need to expand the typical view of the individual, to include relations that extend outside the skin and, at the same time, inquire of larger systems, such as the social, cultural and political, how these make a difference to the relations and materiality that constitute each individual. In terms of a view on development, both perspectives suggest that what is available to be learnt at any moment by an individual must already be within the scope of possibilities for their structure. Given any individual's structure of relations there will be a 'space of the possible' (Davis, 2004, p. 184) in terms of what could be learnt in any moment. It is not possible to know the extent of this 'space' in advance but it is a consequence of the enactivist world-view that any development that occurs has always and already been pre-figured in the structure of that individual.

In terms of analysing and reporting on our research, enactivism leans towards reporting on 'telling' rather than 'typical' examples (Rampton, 2006, p. 408) with the aim of sensitising the reader to new possibilities rather than asserting causal connections. Our approach is one of 'particularization' (Krainer, 2011, p. 52) as we seek 'paradigmatic examples' (Freudenthal, 1981, p. 135) that might allow us to expand Davis's 'space of the possible'. In the next two sections of empirical data we first explain the device that is at the centre of the learning episodes, then the participants and setting, and then present some data from its classroom use.

TouchCounts Tool

TouchCounts is a free, multi-touch, iPad app (www.touchcounts.ca). It has two 'worlds' which have been described in Sinclair and Coles (2017). We focus here on the key features of one of these worlds. In the image below (in the 'Enumerating' world), a child has made five touches on the screen. Each touch provokes *TouchCounts* to say the numeral aloud. The touches can be single or multiple (in which case *TouchCounts* will only say the largest numeral). In Figure 2 'gravity' mode is on, meaning that the discs will 'fall' from where the screen was touched and disappear off the bottom, except for the ones 'caught' by the shelf. After a few seconds (in Figure 2), if there are no other touches, the only disc visible will therefore be the '5'.

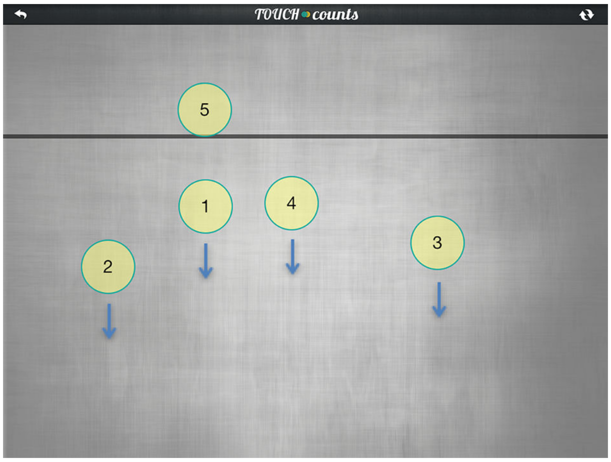

Figure 2. Putting ‘5’ on the shelf.

Initial tasks used in the Enumerating world can include, for example, the challenge of putting (only) ‘5’ on the shelf. In order to complete this, a child needs to make four touches below the shelf and one above. In other words, the task requires awareness that ‘5’ follows ‘4’, an ordinal awareness, with no need for a reference to cardinality. *TouchCounts* brings together the child’s touch, a visual image of a disc created by that touch, a unique symbol on each disc, and a spoken word associated with the symbol. This tool is an example of what we mean by having an interest in children’s learning in ‘novel’ situations. Given the different affordances of *TouchCounts* as a tool compared, for example, to counting objects or bead-strings, we have been exploring what differences might be observed in children’s awareness of place-value, compared to typically observed ‘trajectories’.

Participants and Setting

We draw on two research settings from Canada in exemplifying the use of *TouchCounts*. The first is from a kindergarten classroom in northern British Columbia, in which the teacher and the children were engaged in skip-counting tasks with *TouchCounts*. The second is from an after-school daycare for kindergarten children in which children worked in small groups with the second author and her research team, engaged in tasks designed by Sinclair and Zazkis (2017) to take advantage of the affordances of *TouchCounts*. All the children, in both contexts, were between 5 and 6 years old.

Data Sources and Analysis

We have chosen two illustrative examples of possibilities with *TouchCounts* in the Enumerating world (there is an 'Operating' world as well), one example comes from each research setting described above. We re-analyse part of a dataset that was reported on, with a different focus, in Sinclair and Coles (2017). In both settings video recordings were taken of all sessions using one camera. In the first setting, the teacher held the video camera and recorded sessions involving the use of *TouchCounts* (these typically lasted about 10 minutes). In the second setting, a member of the research team held the camera. We have selected the first example because it shows how the focus on skip counting in *TouchCounts*, seems to suggest alternative pathways that might exist in gaining a sophisticated number sense—particularly in relation to place value, which was not the target goal of any of the tasks used with the children. The second example is meant to highlight the possibility of shifting focus from the cardinal to the ordinal in children's engagement with number, a shift that *TouchCounts* supports through the combination of a tangible and symbolic interface.

Example 1: Ordinal Size

A classroom of kindergarten children sit cross-legged on the carpet. An overhead projector has been connected to an iPad running *TouchCounts*. The teacher begins the lesson by asking children to count by five. They accomplish this by first tapping four fingers simultaneously below the shelf and then one finger above it, which leaves only the numerals 5, 10, 15… on the shelf. These numerals are placed on the shelf in turn, by one child at a time. In the following table, we present the data in a way that can draw attention to the rhythm of the skip counting. We remind the reader that unlike usual skip-couting, in which only the number names "five, ten, fifteen, …" would be heard, in this activity with *TouchCounts*, the following number names are heard: "four, five, nine, ten, fourteen, fifteen, …" This is due to the fact that the first tapping of four fingers below the shelf produces an oral "four" and then the fifth finger above the shelf produces an oral "five".

Child says	Child does	iPad says
	Taps four fingers simultaneously below the shelf	Four
Five	Taps one finger above the shelf	Five
	Taps four fingers simultaneously below the shelf	Nine
Ten	Taps one finger above the shelf	Ten
	Taps four fingers simultaneously below the shelf	Fourteen
Fifteen	Taps one finger above the shelf	Fifteen
	Taps four fingers simultaneously below the shelf	Nineteen
Twenty	Taps one finger above the shelf	Twenty

Although the teacher had only planned to skip count to 25, the children decide to skip count even higher, eventually producing 100. The shelf is therefore full of discs overlapping each other. The teacher wiggles the shelf so that all the multiples of 5 drop off it and disappear off the screen. Since the children want to continue, the teacher obliges and when they get to 125, the children say out loud, in a chanting kind of way, what the next multiple of 5 will be and continue in this way until they get to 200. One student then breaks with the chanting to say something about 200.

Cam:	I thought that two hundred was right after one hundred, but it's not.
Teacher:	No, how far is it away from one hundred?
Cam:	It's, it's, it's one more hundred away.

Since the children were skip-counting, naming one multiple of five after another, they were not being asked to pay attention to the quantity of discs they had produced. Of course, some of these children may have associations between the number of fingers on one of their hands and a sense of the cardinality of '5'. However, having tapped 'four below' and 'one above' the shelf, *TouchCounts* does not present five objects (as would be implied by a cardinal representation) but one disc, labelled '5', and then others labelled '10', '15', etc. The children were presented with an aural and symbolic structure relating to their actions (e.g., the consistent 'five' said at the end of the number name when they reached 25, 35, 45, etc, and the '5' appearing as part of the symbol in the yellow disc of that number), which we conjecture is what enabled the children to begin to chant out the multiples. When the children got to 200, the teacher had not connected the number word they were saying (one hundred and ten, one hundred and fifteen, etc.) to a cardinal quantity. Indeed, Cam noticed something about the relation between 200 and 100 that is not related to cardinality. Instead, Cam observes that in order to get from 100 to 200, one has to undertake the same process of making multiples of 5 that they did to get to 100. This seems to us a deeply temporal relation, connected as it is to the amount of time, of labour, required to make all the multiples of 5 up to 100 and then up to 200. The temporal relation arises from the sequential pronouncements of *TouchCounts*, which says some number names that many of these children would be hearing for the first time and would be associating with their symbolic forms also for the first time.

Typical developmental narratives would suggest that children cannot gain awareness of, for example, the distance between 100 and 200 until they have mastered relations under 10 and then under 20. These children had had no formal schooling and had not yet met addition and subtraction symbols, yet showed evidence of additive relations with numbers over 100. Whatever place value means (and, as suggested above, we feel there is complexity in this) developing a sense that 200 is 100 away from 100 is surely a significant awareness.

Example 2: Making 10

At the after-school daycare, three children are working together with a member of the research team. In their prior visits, the researchers had noticed that the children tended to use just one finger at a time when playing with *TouchCounts*. In order to promote subitising, they decided to encourage the children to use multiple fingers. The first activity for the children was to put just 10 on the shelf, which they did using one finger at a time. At this point, the researcher asked the children to put ten on the shelf using a different strategy.

There is a clear pattern across Auden’s two attempts at this task, of alternating index and either pinkie or middle finger gestures, where the index finger picks out all the odd numbers (see transcript below). This gestural pattern takes Auden beyond ‘9’ on the first attempt, with all discs placed below the shelf, hence losing the ‘10’ off the screen. At the second attempt, Auden disrupts the alternating index finger pattern when he reaches ‘9’ and uses his index finger again to place ‘10’ in the correct position for this task, which is above the shelf. There is evidence, therefore, of his anticipation of the ‘10’ in the count sequence – an ordinal number awareness.

The activity of ‘making 10 in different ways’ is similar to that of partitioning a collection of 10 objects. Partitioning involves paying attention to the cardinality of the subgroups of objects and the overall numerosity (partitioning 10 might involve thinking of 4 and 6 and the fact that the sum is 10). The transcription provided above might suggest that the three children were also thinking of cardinality, we hypothesise that the making of 5, 9 and 10 (for Wesley) and 5, 6, 8, 9 and 10 (for Benford) and the 1, 2, 3, 4, 5, 6, 7, 8, 9 and 10 (for Auden) actually involved ordinal thinking because instead of producing 10 discs, they made their numbers in sequence so that they could get to a 10^{th} tap. In this way, *TouchCounts* provided a context in which a task that is normally done through cardinal thinking can be undertaken in a way that focuses less on size and more on number names and their relations to each other.

Example 2 points to the choices available, as teachers or curriculum designers, in terms of ordinal or cardinal focus for what could seem like equivalent tasks. We do not want to make particular claims this time about what the children did, except to point out that there may be different ways to complete an activity such as ‘making 10 in different ways’ and it seems highly likely to us that ‘normal development’ might look very different in different curricula using different tools (see Sinclair and Pimm, 2015, for further discussion of possibilities of working with *TouchCounts*).

Immediately, Whyles touches:	**TouchCounts says**
with 5 fingers below the shelf,	“five”
then 4 fingers below the shelf,	“nine”
then 1 finger above the shelf.	“ten”
The only thing on the screen is a yellow disc labelled 10 sitting on the shelf.	
Benford goes next, with a blank screen. He touches:	**TouchCounts says**
with 5 fingers below the shelf,	“five”
then 1 finger below the shelf,	“six”
then 2 fingers below the shelf,	“eight”
then 1 finger below the shelf,	“nine”
then puts 1 finger above the shelf	“ten”
The only thing on the screen is a yellow disc labelled ‘10’ sitting on the shelf.	
Auden goes next. He touches:	**TouchCounts says**
with 1 index finger below the shelf,	“one”
then 1 pinkie finger below the shelf,	“two”
then 1 index finger below the shelf,	“three”
then 1 middle finger below the shelf,	“four”
then 1 index finger below the shelf,	“five”
then 1 middle finger below the shelf,	“six”
then 1 index finger below the shelf,	“seven”
then 1 middle finger below the shelf,	“eight”
then 1 index finger below the shelf,	“nine”
then 1 middle finger below the shelf.	“ten”
There is nothing on the screen except the shelf. In particular, there is not the disc labelled 10 sitting on the shelf, which was the goal of the activity.	
Auden presses reset and then touches:	**TouchCounts says**
with 1 index finger	“one”
then 1 middle finger	“two”
then 1 index finger	“three”
then 1 pinkie finger	“four”
then 1 index finger	“five”
then 1 middle finger	“six”
then 1 index finger	“seven”
then 1 middle finger	“eight”
then 1 index finger	“nine”
then puts 1 index finger above the shelf	“ten”

The Gattegno Tens Chart

Our second data source comes from work in primary schools in the UK. The ‘Gattegno tens chart’, or simply ‘Tens chart’ is a visual structuring of our base 10 number system (see Figure 3). Different rows can be displayed, including decimal ones (tenths, hundredths, etc), but the unit row is always included. Like *TouchCounts*, the chart is a non-concrete tool that privileges symbols—acting as a metonymic device rather than a metaphoric one (see Figure 1).

0.001	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009
0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	2	3	4	5	6	7	8	9
10	20	30	40	50	60	70	80	90
100	200	300	400	500	600	700	800	900

Figure 3. An example of a Tens chart.

One way of working with the Tens chart, as a teacher, is by pointing to numerals in turn and asking a class to chant back in unison. Tapping on two numerals (e.g., 50 then 7) the class chants back “fifty-seven”, the number made by the two taps. Students can do the pointing for others to say. The teacher can point to a number and others have to chant back the number: one more, one less, ten more, ten less, a hundred times more, etc. For example, if the teacher points to 20, the students might be asked either to say the next number (21) or the number that is ten more (30) or 100 more (120), each of which will require a very different choreography of tapping by the teacher (21: tap 20 then 1; 30: tap 30; 120: tap 100 then 20). A further challenge might be for a class to skip count using the chart forward, backwards in any number, starting from any number.

Participants and Setting

The first author had been invited by a primary school headteacher to work with a class, in a fairly typical (in terms of prior attainment) rural primary school in the UK, as part of a project linked to the charity “5x5x5=creativity” who place artists (in this case with the first author acting as a ‘mathematician’) into schools to run projects with students and teachers. The class that was the focus of this work comprised thirty students aged 7-8. The first author taught a sequence of four lessons (one per week over a month) making use of the Gattegno chart, with the normal class teacher observing.

Data Sources and Analysis

The first author wrote reflective notes after each session in school and photocopies were taken of all students’ work during these lessons. Coles (2014) has reported on the same sequence of lessons and we draw on examples from the data reported there, which we re-analyse. It is possible to work on multiplicative relations using the chart, e.g., the teacher taps on a number and the class chant back the number ten times bigger, or ten times smaller, or a hundred times; introducing multiplication and division had been chosen by the class teacher as the focus they wanted for the sessions led by the first author. We offer, below, two examples of students’ work. The first (Figure 4) is a typical example and almost any student’s work could have been chosen. The second example (Figure 5) was perhaps the most insightful written comment from the class (in terms of relating to the work being done to wider mathematical structures) although all students annotated their work and most referred to patterns they had noticed or questions that had arisen for them. Analysis was undertaken on the photocopies of student work shown in the Figures and focused on the mathematical concepts being used by students.

Example 1: Inverse Operations

Having set up how to multiply and divide by 10 and 100 on the chart, emphasising the visual way this can be done, Alf (first author) sets a challenge. The students have to choose a number on the chart, go on a journey multiplying or dividing by 10, 100 and to get back to where they started. Alf works through two journeys all together as a class to demonstrate how these are to be written out.

The student from Figure 4 completed three journeys and ticked their own work since they recognized that they got back to where they started (we suggested they tick in this way). There is evidence here of students' confident use of inverse operations and the distinction between directions of movement on the chart, linked to the symbols for ÷ and ×.

Example 2: Going Back in One

All students were able to produce the kinds of journey in Figure 4. In the first lesson taught by Alf, around half the students stayed with the same type of journey done from different starting points, and the other half branched out to try more and different combinations of operations, for example the student who produced the work in Figure 5.

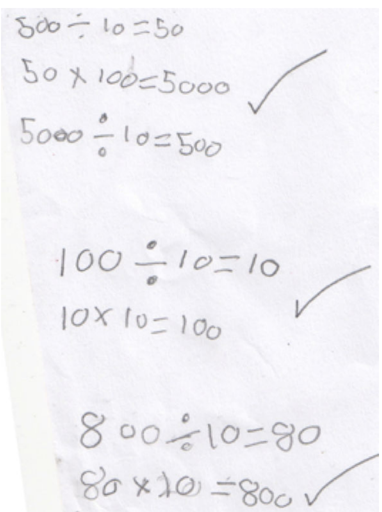

Figure 4. Three 'journeys'.

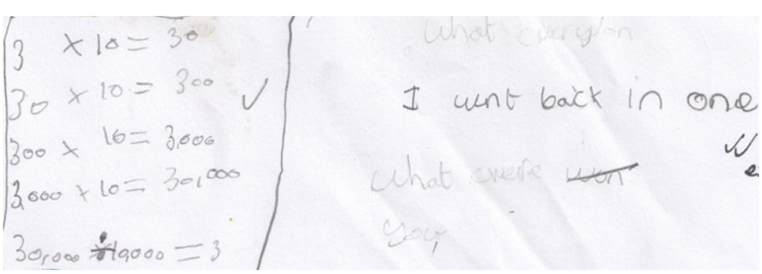

Figure 5. A student extends to division by 10,000: "I went back in one".

There is evidence here of the student extending the pattern of how to divide by 10 and 100, to work out what she must do as the inverse of x10, x10, x10, x10. Alf and the class had not worked on division by numbers greater than 100. As with the *TouchCounts* examples, the student here is working with much bigger numbers than would ever usually be tackled in a Year 3 curriculum. The student appears to have noticed a pattern in how inverse operations work in terms of multiplication and division by powers of ten and extended this awareness to division by 10,000. We are not confident that the student would be able to name the numbers she was working with (e.g., 30,000 as ‘thirty thousand’) much less have a sense of their size. And, although we recognise that this may be disturbing (Coles and Sinclair, 2017) we suggest that this example again points to the sensitivity of the ‘development’ of number sense to tasks, activities, curricula and teachers. If we follow Pimm’s (2018) suggestion that place value does not exist in verbal speech then the fact that students cannot say “thirty thousand” is not important in terms of their understanding of place value. What is more important, in terms of place value, is that students seem to be working with how the symbols for multiplication and division change the ‘place’ of digits in the written form of numbers, in patterned ways.

We want to distinguish what is happening here from a rote memorization of a process (such as ‘adding a zero’ when you multiply by 10, or shifting digits against imagined place value columns). In part, what is different is that the Gattegno chart offers a visual structuring of the number system with many more affordances than memorizing a rule. For example, embedded within the chart are number lines at different scales of magnitude, awareness of which has been linked to developing number sense (Harvey, Klein, Petridou, & Dumoulin, 2013). The chart can be extended to include rows of tenths and hundreds, etc., and in Coles (2014) there is evidence of this class investigating numbers less than 1, seemingly without difficulty or disruption to the sense they were making of numbers greater than 1. We see this as partly explained by having the numbers present, in the image the students are using, which invite exploration and meaning making. Students seem fascinated by big numbers and decimals and the chart can act to open possibilities for *playing* with them. The chart is also powerful as a support for counting (not counting objects, just counting) and skip counting, hence familiarity with the chart brings other pedagogical benefits over memorizing a routine or process that only applies to multiplication by powers of ten.

Discussion and Implications

In this article, we have pointed to the controversies that exist within psychology about whether there are ‘typical’ developmental pathways for children. In particular we highlighted the problematic nature of neurological evidence that reports on adults who have already traversed the typical cardinality focused learning trajectories. We also questioned whether, we could instead begin thinking about ‘variation without a norm’. From a materialist point of view, we might also begin thinking about the imbrication of tools and concepts, and the diversity that would be entailed. In relating this debate to mathematics education, we have particularly focused on the early learning of number, which is the area we have been researching. The approach, within mathematics education, that is most prevalent has been to look for patterns in the achievements of children in their learning of number and to articulate these in terms of a hierarchy. In the past, these results have been interpreted as evidence of developmental pathways. The newer language of learning trajectories attempts to dissociate itself from developmental assumptions and yet appears to end up replicating the description of hierarchies based on particular curriculum approaches, notably with the case of early number. One aim we have had in this writing has been to question assumptions of a normal developmental route and, in this disruption, to see what new questions or possibilities emerge.

The evidence for the existence of an Approximate Number System in humans might be seen as pointing to a 'universal' developmental feature of human learning out of which 'particular' aspects of each culture's naming system emerge. We wonder whether the particular and universal are necessarily so distinct and whether development of particulars might actually shape the development of universals in the context of early number learning.

In order to challenge the predominant view of typical development we looked first at the neuroscience work of Lyons and colleagues. This body of work is establishing the significance, both in learning number and in more sophisticated number use, of ordinal awareness of number symbols. It appears as though different brain patterns are in evidence when we work with number symbols in a manner that is not about relating them to objects, compared to any other kind of number–object comparison or manipulation. The neuroscience findings lead us to suggest that current curricula in the UK and Canada may well over-emphasise cardinality in the early learning of number and hence we have been asking whether, if children are offered images, tasks and technologies that better balance cardinal and ordinal approaches, might we observe different patterns of development from the currently assumed norms?

Our data points to the ways in which children appear able to operate with numbers (hundreds in kindergarten; ten thousands age 7-8) that are well beyond curriculum expectations and also work with concepts (place value, inverse operations) that would appear more abstract, or dependent on full number sense development and where those children have not shown evidence of achievement at 'lower' steps of the list. Our data also shows how equivalent tasks (e.g., partitioning 10) may be approached in either a cardinal-focused or ordinal-focused manner. It is likely that both would be important, but the point we want to raise is that there is a choice here and that depending on what choice is made by teachers, children might 'develop' in different ways.

There is evidence that the ways children work with number is sensitive to curriculum organisation, as well as tool use, and that by carefully studying the entanglement of tools and concepts, we might discover more diversity than we see in 'typical' development. We recognise that this might pose a unique set of problems for policy-makers, teachers and curriculum developers, but that should not constrain our research into children's mathematics learning. Part of what is at stake here is how we view the very concept of number and the extent to which we privilege the cardinal or the ordinal. We see it as an open research question as to what is an adequate balance for children's learning and, for example, whether one side is introduced first or both together?

Through focusing on the detail of the early learning of number our aim was to provide a series of arguments and examples that point to the need for serious debate around the predominant assumption, within mathematics education, of there being typical developmental sequences or trajectories of learning. We want to suggest that 'development' cannot, at this stage of knowledge in our field, be abstracted from the tasks, technologies and concepts at stake (as well as the socio-cultural context). To take seriously the idea of 'variation without a norm' would mean to consider and compare diverse curricula approaches and how learners gain a sense of number, without necessarily trying to find convergence.

We are concerned, in particular, that a view that the learning of mathematics proceeds in a linear or predictable manner (around, say, a particular concept) currently means that a significant minority of students are never offered the more sophisticated uses of mathematics (for example, ordinal, temporal and linguistic ways of working with number) if they cannot first show 'understanding' of earlier steps. This concern seems reasonable, reflecting on the illustrative data offered above, which demonstrates the possibility of conceptualising place value before mastering cardinality. Given the questions that we hope to have raised over the notion of a typical sequence of development, we see no theoretical or empirical justification for the inference from observing that a student cannot do ‘stage *n*’ within a levelled system, to assuming that they therefore could not at this moment do anything at a stage beyond ‘*n*’. Our arguments have concerned number and early learning, however, if they have been effective in raising questions about what typical or normal development is taken to mean, then we hope also they have raised questions about this assumption within mathematics education and psychology more widely.

Notes

i) In the dot test, they varied the way that the dots were shown (such as changing the size of the dots) in order to see how perceptual cues modulated ordinal judgments. This is typical in the neurocognitive research literature on number sense.

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